Scattering, reflection and impedance of waves in chaotic and disordered systems with absorption

Scattering, reflection and impedance of waves in chaotic and disordered systems with absorption

Y V Fyodorov[1], **D V Savin**[2] **and H-J Sommers**[2]

[1] School of Mathematical Sciences, University of Nottingham, Nottingham NG7 2RD, UK
[2] Fachbereich Physik, Universität Duisburg-Essen, 45117 Essen, Germany

Abstract

We review recent progress in analysing wave scattering in systems with both intrinsic chaos and/or disorder and internal losses, when the scattering matrix is no longer unitary. By mapping the problem onto a nonlinear supersymmetric σ-model, we are able to derive closed-form analytic expressions for the distribution of reflection probability in a generic disordered system. One of the most important properties resulting from such an analysis is statistical independence between the phase and the modulus of the reflection amplitude in every perfectly open channel. The developed theory has far-reaching consequences for many quantities of interest, including local Green functions and time delays. In particular, we point out the role played by absorption as a sensitive indicator of mechanisms behind the Anderson localization transition. We also provide a random-matrix-based analysis of S-matrix and impedance correlations for various symmetry classes as well as the distribution of transmitted power for systems with broken time-reversal invariance, completing previous works on the subject. The results can be applied, in particular, to the experimentally accessible impedance and reflection in a microwave or an ultrasonic cavity attached to a system of antennas.

PACS numbers: 05.45.Mt, 24.60.−k, 42.25.Bs, 73.23.−b

1. Introduction

Propagation of electromagnetic or ultrasonic waves in billiards [1], compound-nucleus reactions [2], scattering of light in random media and transport of electrons through quantum dots [3, 4] share at least one feature in common: in all these situations one deals with an open wave-chaotic system studied by means of a scattering experiment, see figure 1 for an illustration. Here, we have a typical transport problem where the fundamental object of interest is the scattering matrix S, which relates linearly the amplitudes of incoming and outgoing fluxes. However, under real laboratory conditions there are a number of different sources which cause
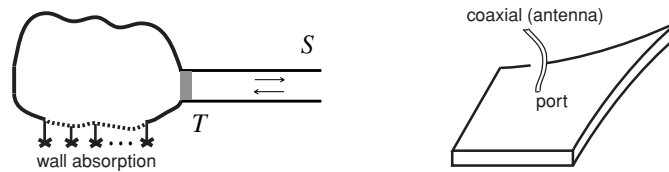
Figure 1. A sketch of a typical experimental setup with microwave billiards. A flat chaotic cavity is fed with microwaves through an attached coaxial cable (i.e. a scattering channel). On average, $1 - T$ part of the incoming flux, where $T \leqslant 1$ is the so-called transmission coefficient, is reflected back directly from the cable–cavity interface (port) without exciting long-lived resonances in the cavity. If the cavity is thin enough then only a transverse electric wave can propagate inside. The electric field has only a vertical component, which is uniform in vertical direction and distributed nontrivially in the plane. Therefore, there is a voltage between plates as well as a current due to the in-plane magnetic field. The impedance is a quantity which relates linearly the port voltage to the port current. Fluctuations of eigenmodes and eigenfrequencies result in fluctuations of the impedance or S-matrix, as the driving frequency or port position is changed.

that a part of the flux gets irreversibly lost or dissolved in the environment. As a result, we encounter absorption and have to handle the S-matrix, which is no longer unitary. Statistics of different scattering observables in the presence of absorption are nowadays under intensive experimental and theoretical investigations, starting from early experiments on reflection and energy correlations of the S-matrix [5, 6]. More recently, total cross-sections [7], distributions of reflection [8] and transmission [9] coefficients as well as that of the complete S-matrix [10] in microwave cavities, properties of resonance widths [11] in such systems at room temperatures, dissipation of ultrasonic energy in elastodynamic billiards [12] and fluctuations in microwave networks [13] became experimentally available. Theoretically, statistics of reflection, delay times and related quantities were considered first in the strong [14] and then weak [15] absorption limits at perfect coupling and very recently at arbitrary absorption and coupling for several symmetry classes [16–24].

Another insight to the same problem comes from looking at it not from the 'outside', but rather from the 'inside'. Then the prime object of interest turns out to be the impedance Z relating linearly voltages to currents at the system input [25, 26], see figure 1. By properly taking into account the wave nature of the current [27, 28] the cavity impedance can be seen as an electromagnetic analogue of Wigner's reaction matrix of the scattering theory. This can be easily understood qualitatively through the well-known equivalence of the two-dimensional Maxwell equations to the Schrödinger equation, the role of the wavefunction being played by the electromagnetic field (the voltage in our case). Then the definition of the impedance becomes formally similar to the definition of the reaction matrix (which relates linearly the scattering wavefunction to its normal derivative on the boundary). The impedance is, therefore, related to the local Green function of the closed cavity and fluctuates strongly due to chaotic internal dynamics.

The imaginary part of the local Green function (which is proportional to the real part of Z) is well known as the local density of states (LDoS) and has a long story of studies in disordered electronic systems, see [29] for a recent review. Actually, a closely related quantity emerges in the context of spectra of complex atoms and molecules where it has the meaning of the total cross-section of indirect photoabsorption [30] (see also [31, 32]). It also appears in studies on spontaneous light emission by atoms placed in chaotic cavities [33]. As to the real part of the Green function, it seems to have no direct physical meaning in mesoscopics while it has the meaning of reactance in electromagnetics, where both real and imaginary parts are experimentally accessible.

In this paper we discuss an approach developed recently by us in short communications [21, 22, 34] which treats both inside and outside aspects of the problem on an equal footing. In this capacity it provides a uniform and deeper understanding of various results on absorptive scattering obtained earlier in [16–18]. In particular, the method allows one to study very efficiently the distribution of the local Green function (complex impedance) at arbitrary absorption and to relate it to that of reflection, thus linking somewhat complementary experiments [10] and [25] together. Although calculations are most explicit and simple for fully chaotic systems, when one can rely upon the random matrix theory (RMT), our method actually has relevance in a much broader context beyond RMT that involves many interesting aspects of disordered mesoscopic systems with absorption, including effects of the Anderson localization. From that point of view, the method opens an attractive possibility to look at some long-standing problems (e.g. statistics of time delays) from a different perspective, see [35] as well as subsections 4.1.6 and 4.1.7.

2. Reflection, time delays and resonance spectrum

In this section, we provide a short description of the scattering approach to the problem. The resonance energy dependence of observables becomes explicit in the well-known Hamiltonian approach to quantum scattering, which was developed first in the context of nuclear physics [2, 36–38] and can be easily adopted for models emerging in quantum chaotic scattering and mesoscopic physics, see e.g. [4, 39–41] for reviews. This framework turns out to be also most suited to take a finite absorption into account. The starting point is the following fundamental relation between the resonance part of the scattering matrix and the Wigner reaction matrix K:

$$S(E) = \frac{1 - \mathrm{i}K(E)}{1 + \mathrm{i}K(E)}, \qquad K(E) = \frac{1}{2}V^\dagger(E - H)^{-1}V. \tag{1}$$

The Hermitian Hamiltonian H of the *closed* system gives rise to N real energy levels (eigenfrequencies). Those are coupled to M continuum channels via the $N \times M$ matrix V of coupling amplitudes V_n^c ($n = 1, \ldots, N$, $c = 1, \ldots, M$), and as a result are converted to N complex *resonances*. To see this, we expand (1) in a Taylor series in K and, after regrouping the terms, bring the resulting expression to another well-known form

$$S(E) = 1 - \mathrm{i}V^\dagger \frac{1}{E - \mathcal{H}_{\mathrm{eff}}} V, \qquad \mathcal{H}_{\mathrm{eff}} = H - \frac{\mathrm{i}}{2}VV^\dagger \tag{2}$$

for the S-matrix. The effective Hamiltonian $\mathcal{H}_{\mathrm{eff}}$ emerging here characterizes the *open* system and is the non-Hermitian counterpart of H. The factorized structure of the anti-Hermitian part is necessary to ensure the unitarity of S. The coupling amplitudes V change very slowly with the energy (far from the channels thresholds) and one can safely consider them to be energy independent. In such a resonance approximation the complex eigenvalues $\mathcal{E}_n = E_n - \frac{\mathrm{i}}{2}\Gamma_n$ of $\mathcal{H}_{\mathrm{eff}}$, with the energies E_n and escape widths $\Gamma_n > 0$, are the only singularities of the S-matrix in the complex energy plane. As required by causality [42], they are located in the lower half plane and correspond to the long-lived resonance states formed on the intermediate stage of a scattering process. The corresponding (left and right) eigenvectors form the so-called bi-orthogonal system. Recent discussion concerning applicability of the effective Hamiltonian approach to potential scattering problems (like those with cavities) can be found in [43–46].

Flux conservation requires $S(E)$ to be unitary at the real values of E. It is useful to define at real Ω the following matrix:

$$R_\Omega(E) \equiv S^\dagger\left(E - \frac{1}{2}\Omega\right) S\left(E + \frac{1}{2}\Omega\right) = 1 + \mathrm{i}\Omega V^\dagger \frac{1}{E - \frac{1}{2}\Omega - \mathcal{H}_{\mathrm{eff}}^\dagger} \frac{1}{E + \frac{1}{2}\Omega - \mathcal{H}_{\mathrm{eff}}} V. \tag{3}$$

The second equality here results from the substitution of (2) and then making use of the identity $\mathrm{i} V V^\dagger + \Omega = \left(E + \frac{1}{2}\Omega - \mathcal{H}_{\mathrm{eff}}\right) - \left(E - \frac{1}{2}\Omega - \mathcal{H}_{\mathrm{eff}}^\dagger\right)$ which leads to cancellations of the cross-terms. Expression (3) tends to unity as $\Omega \to 0$ and it is the Wigner–Smith time-delay matrix [47, 48] $Q(E) = -\mathrm{i} S^\dagger(E) \frac{\mathrm{d}}{\mathrm{d}E} S(E) = -\mathrm{i} \frac{\mathrm{d}}{\mathrm{d}\Omega} R_\Omega(E)\big|_{\Omega=0}$ (we put $\hbar = 1$) which determines the unitarity deficit of R_Ω to the linear order [49]:

$$R_\Omega(E) \simeq 1 + \mathrm{i}\Omega Q(E) + \mathrm{O}(\Omega^2), \qquad Q(E) = V^\dagger \frac{1}{(E - \mathcal{H}_{\mathrm{eff}})^\dagger} \frac{1}{E - \mathcal{H}_{\mathrm{eff}}} V. \tag{4}$$

Such a factorized representation [50] for the time-delay matrix (which no longer contains the energy derivative) is a consequence of the resonance approximation considered. It serves to make a connection of time delays to the resonance spectrum most explicit. The matrix element $Q^{cc'} = (b^\dagger b)^{cc'}$ may be physically interpreted as the scalar product (or 'overlap') between the internal parts $b^c = (E - \mathcal{H}_{\mathrm{eff}})^{-1} V^c$ of the scattering wavefunctions for waves incident in the channels c and c', respectively [50]. In particular, the mean time delay in the channel c given by the diagonal element Q^{cc} coincides in such an approximation with the dwell time given by the norm of b^c. One should distinguish generally between Q^{cc} and the so-called *proper* time delays q_c (eigenvalues of Q). Taking their sum, one comes to a weighted mean time-delay characteristic $Q_w = \frac{1}{M} \operatorname{tr} Q = \frac{1}{M} \sum_c Q^{cc} = \frac{1}{M} \sum_c q_c$, the so-called Wigner's time delay which is known, see e.g. [40, 49, 51], to be determined by the energy derivative of the total scattering phase shift: $Q_w = -\frac{\mathrm{i}}{M} \frac{\partial}{\partial E} \log \det S$. Diverse aspects of delay times in quantum chaotic scattering [40] as well as in a general quantum mechanical context [52, 53] can be found in the cited literature and references therein.

As is well known, statistics of spectra of *closed* quantum systems with chaotic classical counterparts are to a large extent universal and independent of their microscopic nature. This remarkable universality provides one with a possibility of using the RMT [54] for an adequate description of many physical properties of such systems [55]. According to the general paradigm we replace the actual Hermitian part H of the effective Hamiltonian (2) with a random Hermitian matrix H taken from one of the three canonical Wigner–Dyson's ensembles labelled by the symmetry index β according to the symmetry of the system under consideration. The Gaussian orthogonal (GOE, $\beta = 1$ and H real symmetric) and unitary (GUE, $\beta = 2$ and H Hermitian) ensembles stand for systems with preserved or fully broken time-reversal symmetry (TRS), respectively. The remaining Gaussian symplectic ensemble (GSE, $\beta = 4$ and H self-dual quaternion) is relevant for description of time-reversal systems with strong spin–orbit scattering. The limit of large $N \to \infty$ is supposed to be finally taken. Then eigenvalues are distributed on the finite interval according to Wigner's semicircle law, which determines locally the mean level spacing Δ. The most appealing feature of the RMT approach is that quantities related to spectral fluctuations when expressed in units of Δ ('unfolding') do not depend on microscopic details (i.e. the particular form of the distribution of H or the profile of Δ) and become uniform throughout the whole spectrum [55]. For practical reasons we thus restrict ourselves to considering fluctuations at the centre of the spectrum ($E = 0$) only. Similarly, the results turn out to be also independent of particular statistical assumptions on coupling amplitudes V_n^c as long as $M \ll N$ [49, 56]. The amplitudes may be chosen as fixed [2] or random [38] variables and enter final expressions only in combinations known as transmission coefficients (also sometimes called sticking probabilities)

$$T_c \equiv 1 - |\overline{S^{cc}}|^2 = \frac{4\kappa_c}{(1 + \kappa_c)^2}, \qquad \kappa_c = \frac{\pi \|V^c\|^2}{2N\Delta}, \tag{5}$$

where $\overline{S^{cc}}$ stands for the average (optical) S-matrix. The transmission coefficients are assumed

to be input parameters of the theory, the cases $T_c \ll 1$ and $T_c = 1$ corresponding to an almost closed or perfectly open channel 'c', respectively.

Absorption is usually seen as a dissipation process, which evolves exponentially in time. Strictly speaking, different spectral components of the field may have different dissipation rates. However, frequently this rather weak energy dependence can easily be neglected as long as *local* fluctuations on much finer energy scale $\sim \Delta$ are considered. As a result, all the resonances acquire one and the same absorption width $\Gamma > 0$ additionally to their escape widths Γ_n. The dimensionless phenomenological parameter $\gamma \equiv 2\pi\Gamma/\Delta$ characterizes then the absorption strength, with $\gamma \ll 1$ or $\gamma \gg 1$ corresponding to the weak or strong absorption limit, respectively. Microscopically, it can be modelled by means of a huge number of weakly open parasitic channels [6, 57] or by coupling to a very complicated background with almost continuous spectrum [16], see also [58]. In microwave billiards such an approximation is frequently very good to account for uniform Ohmic losses which happen everywhere in non-perfectly conducting walls. However, in some experimentally relevant situations such as, e.g., complex reverberant structures [59] or even microwave cavities at room temperature [11, 46] an approximation of uniform absorption may break down, and one should take into account instead localized-in-space losses which will result in different broadenings of different modes. The latter are easily incorporated in the model by treating them as if induced by additional scattering channels, see e.g. [19]. An alternative scheme of treating localized-in-space surface absorption is discussed in [23]. (Discussion of a formal theory of scattering for complex absorbing potentials can be found in [60].)

Operationally, the *uniform* absorption can equivalently be taken into account by a purely imaginary shift of the scattering energy $E \rightarrow E + \frac{i}{2}\Gamma \equiv E_\gamma$, so that the S-matrix $S_\gamma(E) \equiv S(E_\gamma)$ becomes subunitary. The reflection matrix $R_\gamma = S_\gamma^\dagger S_\gamma$ provides then a natural measure of the mismatch between incoming and outgoing fluxes. It can be obtained from R_Ω (3) by analytic continuation in Ω from a real to the purely imaginary value $\Omega = i\Gamma$, yielding [16]

$$R_\gamma(E) = R_{\Omega=i\Gamma}(E) = 1 - \Gamma Q_\gamma(E), \qquad Q_\gamma(E) \equiv V^\dagger \frac{1}{(E_\gamma - \mathcal{H}_{\text{eff}})^\dagger} \frac{1}{E_\gamma - \mathcal{H}_{\text{eff}}} V. \qquad (6)$$

This representation is valid at arbitrary value of Γ. In the limit of small Γ one can neglect the difference between Q_γ and Q, resulting in the approximate expression [15, 61] $R_\gamma \simeq 1 - \Gamma Q$ following from (4). It is therefore tempting to keep for Q_γ the meaning of the time-delay matrix at finite absorption as well (see, however, discussion in [16]). By construction, R_γ is a Hermitian matrix, its positive *reflection eigenvalues* $r_c = 1 - \Gamma q_c \leqslant 1$ are related to proper time delays q_c (at finite absorption).

Considering quantum scattering with no internal dissipation, $\Gamma = 0$, the S-matrix unitarity ensures that the reflection coefficient $R_c = |S^{cc}|^2$ in any given channel is simply related to the quantum mechanical probability to exit via any of the remaining channels, known as the transmission probability: $R_c = 1 - \sum_{b \neq c} |S^{bc}|^2$. For an absorptive system the last equality is violated, but still the quantity $\tau_c = 1 - R_c$ can be interpreted as the quantum mechanical probability that a particle entering via a given channel never exits through the same channel. Hence, it was suggested to call τ_c the *probability of no return* (PNR) [17]. In the particular case of a single open channel fluctuations of the PNR τ arise solely due to absorption (neglecting dissipation trivially results in $\tau \equiv 0$). At weak absorption $\tau \approx \Gamma q$, so that the PNR is just simply proportional to the time delay.

Last but not least, the matrix $Z \equiv iK(E_\gamma)$ has the meaning of the normalized cavity impedance in such a setting, see [27, 28] for further details.

3. Correlation functions

Any observable in chaotic resonance scattering exhibits strong fluctuations over a smooth background as the scattering energy or other external parameters are varied. Usually these variations occur on two essentially different energy scales. This fact is conventionally taken into account by decomposing fluctuating quantities into a mean part and a fluctuating part, the former being understood as the result of spectral or (assumed to be equivalent) ensemble average $\langle \cdots \rangle$. In this section we consider statistics as determined by a two-point correlation function of the fluctuating parts (frequently called a 'connected' correlation function): $\langle AB \rangle_{\mathrm{conn}} = \langle AB \rangle - \langle A \rangle \langle B \rangle$. We restrict ourselves below to the cases of preserved ($\beta = 1$) and broken ($\beta = 2$) TRS (the symplectic case $\beta = 4$ proceeds along the same lines).

3.1. Impedance

Let us start with considering the simplest case of the impedance correlations. The problem can be fully reduced to that of spectral correlations determined by the two-point cluster function $Y_{2,\beta}(\omega) = \delta(\omega) - \Delta^2 \langle \rho(E_1) \rho(E_2) \rangle_{\mathrm{conn}}$, where $\omega = (E_2 - E_1)/\Delta$ and $\rho(E)$ being the level density. It is easy to satisfy oneself that for the mean impedance at $E = 0$ holds $\langle Z^{ab} \rangle = \kappa_a \delta^{ab}$. To calculate the energy correlation function $C_Z^{abcd}(\omega) \equiv \langle Z^{ab*}(E_1) Z^{cd}(E_2) \rangle_{\mathrm{conn}}$, it is instructive to write $Z^{ab}(E)$ in the eigenbasis of the closed system: $Z^{ab}(E) = \frac{1}{2} \sum_n v_n^{a*} v_n^b / (E - E_n + \frac{i}{2}\Gamma)$. A rotation that diagonalizes the (random) Hamiltonian matrix H transforms the (fixed) coupling amplitudes V_n^a to Gaussian-distributed random coupling amplitudes v_n^a with zero mean and covariances $\langle v_n^{a*} v_m^b \rangle = (2\kappa_a \Delta/\pi) \delta^{ab} \delta_{nm}$. In such a representation the energy correlation function acquires the following form:

$$C_Z^{abcd}(\omega) = \sum_{n,m} \frac{1}{4} \langle v_n^a v_n^{b*} v_m^{c*} v_m^d \rangle \left\langle \frac{1}{E_1 - E_n - \frac{i}{2}\Gamma} \frac{1}{E_2 - E_m + \frac{i}{2}\Gamma} \right\rangle_{\mathrm{conn}} \tag{7}$$

and the averaging over coupling amplitudes and that over the spectrum can be done independently. The Gaussian statistics of v results in

$$\frac{1}{4} \left(\frac{\pi}{\Delta} \right)^2 \langle v_n^a v_n^{b*} v_m^{c*} v_m^d \rangle = \kappa_a \kappa_c \delta^{ab} \delta^{cd} + \kappa_a \kappa_b (\delta^{ac} \delta^{bd} + \delta_{1\beta} \delta^{ad} \delta^{bc}) \delta_{nm}, \tag{8}$$

where $\delta_{1\beta}$ term accounts for the presence of TRS, when all v_n^a are real and Z is symmetric. It is useful to represent the spectral correlation function in a form of the Fourier integral $\int_0^\infty dt_1 \int_0^\infty dt_2 \, e^{-\Gamma(t_1+t_2)/2} e^{iE(t_2-t_1)} e^{i(E_2-E_1)(t_1+t_2)/2} \langle e^{i(E_n t_1 - E_m t_2)} \rangle_{\mathrm{conn}}$. Due to the uniformity of local fluctuations in the bulk of the spectrum, one can integrate additionally over the position E of the mean energy: $\int \frac{dE}{N\Delta} e^{iE(t_2-t_1)} = \frac{1}{N} \delta \left(\frac{t_2-t_1}{t_H} \right)$, where $t_H \equiv 2\pi/\Delta$ is the Heisenberg time. It is natural therefore to measure the time in units of t_H. From the known RMT spectral fluctuations one also has for $n \neq m$ $(1-N) \langle e^{2\pi i (E_n - E_m)t/\Delta} \rangle_{\mathrm{conn}} = b_{2,\beta}(t)$, where $b_{2,\beta}(t)$ is the spectral form factor defined through the Fourier transform of $Y_{2,\beta}(\omega) = \int_{-\infty}^\infty dt \, e^{2\pi i \omega t} b_{2,\beta}(t)$ [54, 55]:

$$b_{2,\beta=1}(t) = [1 - 2t + t \log(1 + 2t)]\Theta(1-t) + \left[t \log \left(\frac{2t+1}{2t-1} \right) - 1 \right] \Theta(t-1) \tag{9a}$$

$$b_{2,\beta=2}(t) = (1-t)\Theta(1-t) \tag{9b}$$

at $t > 0$ and $b_{2,\beta}(-t) = b_{2,\beta}(t)$. Combining all these together, we arrive finally at

$$C_Z^{abcd}(\omega) = \int_0^\infty dt \, e^{2\pi i \omega t} \hat{C}_Z^{abcd}(t) \tag{10a}$$

$$\hat{C}_Z^{abcd}(t) = 4\,\mathrm{e}^{-\gamma t}[\kappa_a \kappa_c[1 - b_{2,\beta}(t)]\delta^{ab}\delta^{cd} + \kappa_a \kappa_b(\delta^{ac}\delta^{bd} + \delta_{1\beta}\delta^{ad}\delta^{bc})]. \quad (10b)$$

Similar in spirit calculations were done in a context of reverberation in complex structures in [59, 62] and in a context of chaotic photodissociation in [63, 64].

The form factor (10b) is simply related to that of K-matrix elements at zero absorption as $\hat{C}_Z^{abcd}(t) = \mathrm{e}^{-\gamma t}\hat{C}_K^{abcd}(t)$. Such a relationship between the corresponding form factors with and without absorption is generally valid for any correlation function which may be reduced to the two-point correlation function of resolvents (see [7] and the discussion below, e.g., for the case of the S-matrix). This can be easily understood as a result of the analytic continuation $2\pi\omega \rightarrow 2\pi\omega + \mathrm{i}\gamma$ of the energy difference ω reflecting switching on the absorption (see the previous section).

The obtained expressions describe a gradual loss of correlations in Z-matrix elements as the energy difference grows; generally, $C_Z(\omega \rightarrow \infty) \rightarrow 0$. At $\omega = 0$, (10a) provides us with impedance variances $C_Z^{abab}(0) = \mathrm{var}(Z^{ab}) \equiv \langle |Z^{ab}|^2 \rangle - |\langle Z^{ab} \rangle|^2$, which were recently studied experimentally in [65]. In analogy with the so-called elastic enhancement factor considered frequently in nuclear physics [66], one can define the following ratio of variances in reflection ($a = b$) to that in transmission ($a \neq b$):

$$W_{Z,\beta} \equiv \frac{\sqrt{\mathrm{var}(Z^{aa})\mathrm{var}(Z^{bb})}}{\mathrm{var}(Z^{ab})} = 2 + \delta_{1\beta} - \int_0^\infty \mathrm{d}s\,\mathrm{e}^{-s} b_{2,\beta}\left(\frac{s}{\gamma}\right) \quad (11)$$

where the second equality follows easily from (10b) (note that the coupling constants $\kappa_{a,b}$ are mutually cancelled here). Making use of $b_{2,\beta}(\infty) = 0$ and $b_{2,\beta}(0) = 1$, one can readily find $W_{Z,\beta}$ in the limiting cases of weak or strong absorption as

$$W_{Z,\beta} = \begin{cases} 2 + \delta_{1\beta} & \text{at } \gamma \ll 1 \\ 1 + \delta_{1\beta} & \text{at } \gamma \gg 1. \end{cases} \quad (12)$$

$W_{Z,\beta}$ decays monotonically as absorption grows. In the case of unitary symmetry, (9b) and (11) yield explicitly $W_{Z,2} = 1 + \frac{1}{\gamma}(1 - \mathrm{e}^{-\gamma})$ in agreement with [65]. It is hardly possible to get a simple explicit expression at finite γ in the case of orthogonal symmetry. However, a reasonable approximation can be found if one notices that the integration in (11) is determined mainly by the region $s \leqslant 1$, so that one can approximate $b_{2,1}(s) \approx (1 - 2s + 2s^2)\Theta(1 - s)$ through its Taylor expansion. Performing the integration, one arrives at $W_{Z,1} \approx 3 - \gamma^{-2}[(4 + \gamma^2)(1 - \mathrm{e}^{-\gamma}) - 2\gamma(1 + \mathrm{e}^{-\gamma})]$, which turns out to be a good approximation to the exact answer at moderately strong absorption (deviations are seen numerically only at $\gamma \sim 1$).

3.2. Scattering matrix

The energy correlation function of the scattering matrix elements

$$C_S^{abcd}(\omega) \equiv \langle S_\gamma^{ab*}(E_1) S_\gamma^{cd}(E_2) \rangle_{\mathrm{conn}} = \int_0^\infty \mathrm{d}t\,\mathrm{e}^{2\pi\mathrm{i}\omega t}\hat{C}_S^{abcd}(t) \quad (13)$$

is a much more complicated object for an analytical consideration as compared to (7). The reason becomes clear if one considers again the pole representation of the S-matrix which follows from (2): $S^{ab}(E) = \delta^{ab} - \mathrm{i}\sum w_n^a \tilde{w}_n^b / (E - \mathcal{E}_n)$. Due to unitarity constraints imposed on S at real E the residues and complex resonance energies develop nontrivial correlations [38]. Although results on statistics of resonances in the complex plane [40, 67–69] as well as the corresponding residues [69, 70] became recently available in some particular cases, this knowledge is insufficient as yet for calculating S-matrix correlations in general. The separation like (7) into a 'coupling' and 'spectral' average is no longer possible and can be done only

by involving some approximations [71]. The powerful supersymmetry method [2, 72] turns out to be an appropriate technique to perform the statistical average in this case. In their seminal paper [2], Verbaarschot, Weidenmüller and Zirnbauer performed the exact calculation of (13) at arbitrary transmission coefficients (and zero absorption) in the case of orthogonal symmetry. This finding was later adopted [7] to include absorption. The corresponding exact result for unitary symmetry has been recently presented by us in [34] (see also [73] concerning the S-matrix variance in the GOE–GUE crossover at perfect coupling) and is discussed below.

The calculation proceeds along the same lines as in [2]. The final expression for both the connected correlation function and its form factor (13) can be equally represented as follows:

$$C_S^{abcd} = \delta^{ab}\delta^{cd}T_a T_c\sqrt{(1-T_a)(1-T_c)}\,J_{ac} + (\delta^{ac}\delta^{bd} + \delta_{1\beta}\delta^{ad}\delta^{bc})T_a T_b P_{ab}. \qquad (14)$$

Here, the $\delta_{1\beta}$ term accounts trivially for the symmetry property $S^{ab} = S^{ba}$ in the presence of TRS. J_{ac} and P_{ab} defined below are some functions (of the energy difference ω or the time t), which depend also on TRS, coupling and absorption but already in a nontrivial way. As a result, the elastic enhancement factor $W_{S,\beta} \equiv \sqrt{\mathrm{var}(S^{aa})\mathrm{var}(S^{bb})}/\mathrm{var}(S^{ab})$ is generally a complicated function of all these parameters, in contrast to (11). In the particular case of perfect coupling, all $T_c = 1$, one has obviously from (14) that $W_{S,\beta} = 1 + \delta_{1\beta}$ at any absorption strength.

We consider first expression (14) in the energy domain at real ω (no absorption). The functions $J_{ac}(\omega)$ and $P_{ab}(\omega)$ can be generally written as certain expectation values in the field theory (nonlinear zero-dimensional supersymmetric σ-model) whose explicit representations depend on the symmetry case considered; we refer the reader to [2, 74] for general discussion. In the $\beta = 1$ case of orthogonal symmetry the well-known result of [2] reads:

$$J_{ac}(\omega) = \left\langle \left(\frac{\mu_1}{1+T_a\mu_1} + \frac{\mu_2}{1+T_a\mu_2} + \frac{\mu_0}{1-T_a\mu_0} \right) \right.$$
$$\left. \times \left(\frac{\mu_1}{1+T_c\mu_1} + \frac{\mu_2}{1+T_c\mu_2} + \frac{\mu_0}{1-T_c\mu_0} \right) \mathcal{F}_M \right\rangle_\mu \qquad (15a)$$

$$P_{ab}(\omega) = \left\langle \left(\frac{\mu_1(1+\mu_1)}{(1+T_a\mu_1)(1+T_b\mu_1)} + \frac{\mu_2(1+\mu_2)}{(1+T_a\mu_2)(1+T_b\mu_2)} \right. \right.$$
$$\left. \left. + \frac{\mu_0(1-\mu_0)}{(1-T_a\mu_0)(1-T_b\mu_0)} \right) \mathcal{F}_M \right\rangle_\mu \qquad (15b)$$

with $\mathcal{F}_M = \prod_c \left[\frac{(1-T_c\mu_0)^2}{(1+T_c\mu_1)(1+T_c\mu_2)} \right]^{1/2}$ being the so-called channel factor, which accounts for system openness, and $\langle (\cdots) \rangle_\mu$ is to be understood explicitly as

$$\frac{1}{8}\int_0^\infty d\mu_1 \int_0^\infty d\mu_2 \int_0^1 d\mu_0 \frac{(1-\mu_0)\mu_0|\mu_1-\mu_2|e^{i\pi\omega(\mu_1+\mu_2+2\mu_0)}}{[(1+\mu_1)\mu_1(1+\mu_2)\mu_2]^{1/2}(\mu_0+\mu_1)^2(\mu_0+\mu_2)^2}(\cdots). \qquad (16)$$

In the $\beta = 2$ case of unitary symmetry we have found [34] that

$$J_{ac}(\omega) = \left\langle \left(\frac{\mu_1}{1+T_a\mu_1} + \frac{\mu_0}{1-T_a\mu_0} \right) \left(\frac{\mu_1}{1+T_c\mu_1} + \frac{\mu_0}{1-T_c\mu_0} \right) \mathcal{F}_M \right\rangle_\mu \qquad (17a)$$

$$P_{ab}(\omega) = \left\langle \left(\frac{\mu_1(1+\mu_1)}{(1+T_a\mu_1)(1+T_b\mu_1)} + \frac{\mu_0(1-\mu_0)}{(1-T_a\mu_0)(1-T_b\mu_0)} \right) \mathcal{F}_M \right\rangle_\mu \qquad (17b)$$

with the channel factor $\mathcal{F}_M = \prod_c \frac{1-T_c\mu_0}{1+T_c\mu_1}$ and the corresponding integration being

$$\int_0^\infty d\mu_1 \int_0^1 d\mu_0 (\mu_1+\mu_0)^{-2} e^{i2\pi\omega(\mu_1+\mu_0)}(\cdots). \qquad (18)$$

In the important particular case of the single open channel (elastic scattering), the general expression for the $\beta = 2$ case simplifies further to

$$\langle S^*(E_1) S(E_2) \rangle_{\text{conn}} = T^2 \int_0^\infty \mathrm{d}\mu_1 \int_0^1 \frac{\mathrm{d}\mu_0}{\mu_1 + \mu_0} \frac{1 + (2 - T)\mu_1}{(1 + T\mu_1)^3} \, \mathrm{e}^{\mathrm{i}2\pi\omega(\mu_1 + \mu_0)}. \tag{19}$$

Finally, putting above $\omega \to \omega + \mathrm{i}\gamma/2\pi$ accounts for the finite absorption strength γ.

To consider (14) in the time domain, i.e. the form factor $\hat{C}_S^{abcd}(t)$, we notice that the variable $t = \frac{1}{2}(\mu_1 + \mu_2 + 2\mu_0)$ for $\beta = 1$ or $t = \mu_1 + \mu_0$ for $\beta = 2$ plays the role of the dimensionless time (in units of t_H). The corresponding expressions for $\hat{P}_{ab}(t)$ and $\hat{J}_{ac}(t)$ can be investigated using the methods developed in [41, 66, 71]. For orthogonal symmetry it was done in [7], where the overall decaying factor $\mathrm{e}^{-\gamma t}$ due to absorption was also confirmed by comparison to the experimental result for the form factor measured in microwave cavities. It is useful for the qualitative description to note that $\hat{P}_{ab}(t)$ and $2\hat{J}_{ac}(t)$ are quite similar to the 'norm leakage' decay function [75] and the form factor of Wigner's time delays [49], respectively (they would coincide exactly at $\gamma = 0$, if we put $T_{a,b,c} = 0$ appearing explicitly in denominators above). Then one can follow analysis performed there, see also [41], to find $\hat{P}_{ab}(t) \approx \mathrm{e}^{-\gamma t}$ and $\hat{J}_{ac}(t) \approx (2t/\beta)\, \mathrm{e}^{-\gamma t}$ as exact asymptotic at small times [71], while they both become proportional to $\mathrm{e}^{-\gamma t} t^{-M\beta/2 - 2}$ at large times.

Such a power law is characteristic for open systems [39, 41, 75]. Physically, it results from width fluctuations, which diminish as the number M of open channels grows [40, 75]. In the limiting case $M \to \infty$ and $T_c \to 0$, all the resonances acquire just the same escape width $\sum_c T_c$ (in units of t_H^{-1}), which is often called Weisskopf's width [76], so that the total width is $\gamma_T = \sum_c T_c + \gamma$. Then there occur further simplifications, $\hat{P}_{ab}(t) = \mathrm{e}^{-\gamma_T t}$ and $\hat{J}_{ab}(t) = [1 - b_{2,\beta}(t)]\, \mathrm{e}^{-\gamma_T t}$, which finally result in

$$C_S^{abcd}(\omega) = \frac{(\delta^{ac}\delta^{bd} + \delta_{1\beta}\delta^{ad}\delta^{bc})T_a T_b}{\gamma_T - 2\pi\mathrm{i}\omega} + \delta^{ab}\delta^{cd} T_a T_c \int_0^\infty \mathrm{d}t\,[1 - b_{2,\beta}(t)]\, \mathrm{e}^{-(\gamma_T - 2\pi\mathrm{i}\omega)t}. \tag{20}$$

For the case of $\beta = 1$ this result (at zero absorption) was obtained earlier by Verbaarschot [66]. In the limit considered, expression (20) is very similar to (10a) and (10b), so that the enhancement factor $W_{S,\beta}$ is given by the same (11) where γ is to be substituted with γ_T. At $\gamma_T \gg 1$ (large resonance overlapping or strong absorption, or both) the dominating term in (20) is the first one, which is known as the Hauser–Feshbash relation [77], see [78–80] for discussion. Then $W_{S,\beta} = 2/\beta = W_{Z,\beta}$ that can be also understood as the consequence of the Gaussian statistics of S (as well as of Z) in the limit of strong absorption [80].

3.3. Reflection

In this subsection we consider fluctuations in the weighted-mean reflection coefficient defined as $r = \frac{1}{M} \mathrm{tr}\, R$. Its average value $\langle r \rangle$ was recently calculated in [16], where the exact result was found to be

$$\langle r \rangle = 1 - \frac{\gamma}{M}\left[1 - \gamma \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\gamma t} P(t)\right] \tag{21}$$

relating $\langle r \rangle$ to the so-called norm-leakage function $P(t)$ introduced in [75]. The actual value of $\langle r \rangle$ changes with growing absorption between $1 - \frac{\gamma}{M}$ at $\gamma \ll 1$ and $1 - \frac{1}{M}\sum_c T_c$ at $\gamma \gg M$. For the sake of simplicity we consider the case of M equivalent channels below, all $T_c = T$. For large $M \gg 1$ the widths cease to fluctuate, yielding $P(t) = \mathrm{e}^{-MTt}$, so that (21) results in $\langle r \rangle = (MT + \gamma(1 - T))/(MT + \gamma)$ [18].

The correlation function $C_R(\omega) = \langle r(E_1) r(E_2) \rangle_{\text{conn}} = \Gamma^2 \langle Q_w(E_1) Q_w(E_2) \rangle_{\text{conn}}$ is simply related by virtue of (6) to that of Wigner's time delay $Q_w = \frac{1}{M} \mathrm{tr}\, Q_\gamma$ (at finite Γ).

Amounting to a four-point correlation function of resolvents, its evaluation is generally beyond the present day state of art in the supersymmetry method. However, by approximating $Q_w \approx \frac{1}{M} \sum_n \Gamma_n / [(E - E_n)^2 + \frac{1}{4}(\Gamma_n + \Gamma)^2]$, cf (6), and making use of the rescaled Breit–Wigner approximation (RBWA) of Gorin and Seligman [71], one can find an expression which is supposed [7] to work reasonably well at finite absorption. The final result can be naturally represented in the form of the Fourier-integral

$$C_R(\omega) = \frac{2\gamma^2}{M^2} \int_0^\infty dt \cos(2\pi\omega t) e^{-\gamma t} F_R(t) \tag{22}$$

where the form factor (we have set apart in (22) the trivial absorbing factor $e^{-\gamma t}$) is

$$F_R(t) = \frac{c_{2,\beta}(t)}{(1 + 2Tt/\beta)^{M\beta/2}} - \frac{b_{2,\beta}(t)c_{1,\beta}\left(\frac{t}{2}\right)^2}{(1 + Tt/\beta)^{M\beta}} \tag{23}$$

with the function $c_{n,\beta}(t)$, $n = 1, 2$, defined as follows:

$$c_{n,\beta}(t) = \prod_{l=0}^{n-1} \left(\frac{1}{2}M\beta + l\right) \int_0^\infty dx \frac{x^{n-1} e^{-(\beta\gamma/2T)(1+2Tt/\beta)x}}{(1 + x)^{M\beta/2+n}}. \tag{24}$$

It is instructive to consider the limiting cases of weak and strong absorption in more detail. At $\gamma \ll T \leqslant 1$, the γ-dependence and thus t-dependence of $c_{n,\beta}(t)$ is very weak, so that $c_{n,\beta}(t) \approx 1$ for $t \ll 1/\gamma$. As a result, the form factor reduces to

$$F_R(t) = (1 + 2Tt/\beta)^{-M\beta/2} - b_{2,\beta}(t)(1 + Tt/\beta)^{-M\beta} \tag{25}$$

which is, as expected, just the form factor of Wigner's time-delay correlation function at *zero* absorption calculated in the same scheme RBWA. In the opposite limiting case $\gamma \gg T$, the x-integration in $c_{n,\beta}(t)$ is determined mainly by the region of small $x \leqslant T/\gamma \ll 1$, which readily yields

$$F_R(t) = \left(\frac{MT}{\gamma}\right)^2 \left[\frac{1 + 2/M\beta}{(1 + 2Tt/\beta)^{M\beta/2+2}} - \frac{b_{2,\beta}(t)}{(1 + Tt/\beta)^{M\beta+2}}\right]. \tag{26}$$

Comparing the large time asymptotic of $F_R(t)$, one sees that a *crossover* from $t^{-M\beta/2}$ to $t^{-M\beta/2-2}$ behaviour occurs at $\gamma \sim T$, as we go from weak to strong absorption regime. This behaviuour is indicative of an additional decay of correlations induced by strong absorption on top of the pure exponential one. It is most pronounced for a single-channel cavity with TRS, corresponding to changing from $t^{-1/2}$ to $t^{-5/2}$.

4. Distribution functions

4.1. Single-channel scattering

4.1.1. Reflection coefficient and the local Green function.
It is quite clear that the joint distribution of the real and imaginary parts of K

$$\mathcal{P}(u, v) = \langle \delta(u - \text{Re}\, K)\delta(v + \text{Im}\, K)\rangle \tag{27}$$

determines fully the statistics of the impedance and the S-matrix. By definition (1), $K = \kappa(N\Delta/\pi)G_{ii}$ is related to the diagonal element $G_{ii} = \langle i|(E_\gamma - H)^{-1}|i\rangle$ of the Green function of the closed system in the position representation, taken at the position i of channel attachment. The RMT assumption for H implies however that the basis of representation may be chosen arbitrary. One studies, therefore, statistics of the dimensionless local Green function $K \equiv u - iv$, with $v > 0$ being the local density of states (LDoS). We consider first the case

of *perfect* coupling $\kappa = 1$ ($T = 1$). In particular, this choice ensures the normalization of the mean LDoS such that $\langle v \rangle = 1$.

In this case, the following general form of the distribution function:

$$\mathcal{P}(u, v) = \frac{1}{2\pi v^2} P_0(x), \qquad x = \frac{u^2 + v^2 + 1}{2v} > 1 \tag{28}$$

may be easily established [21]. It initially emerged in [81, 82] in the course of tedious calculations for the $\beta = 2$ symmetry class, but neither origin nor generality of such a form was appreciated. Actually, the validity of the form (28) follows directly from exploiting the two following fundamental properties of the S-matrix statistics at perfect coupling: (i) the uniform distribution of the scattering phase $\theta \in (0, 2\pi)$; and (ii) the statistical independence of θ and the S-matrix modulus. In the cases of chaotic systems with pure symmetries, both these properties can be verified making use of methods of [57]. In this paper we are proving (28) directly by an alternative, more powerful method [22]. It holds much beyond the universal RMT regime for a very broad class of disordered Hamiltonians, in particular for the case when localization effects already play an important role. Moreover, this method will help us to verify that (28) holds in the crossover regime between the pure ensembles. For the time being the uniformity of the scattering phase distribution will be taken for granted.

Substituting $K = u - \mathrm{i}v$ into (1), we can immediately infer that the variable x is directly related to the reflection coefficient, parametrizing it in the following way:

$$S_\gamma \equiv \sqrt{r}\, \mathrm{e}^{\mathrm{i}\theta}, \qquad r = \frac{x - 1}{x + 1}, \qquad \cot \theta = \frac{u^2 + v^2 - 1}{2u}. \tag{29}$$

The joint distribution $\mathcal{P}(u, v)$ of u and v and that $P(x, \theta)$ of x and θ are related by means of the Jacobian $\left| \frac{\partial(u,v)}{\partial(x,\theta)} \right| = v^{-2}$ as can be verified by a straightforward calculation. Finally, $P(x, \theta) = \frac{1}{2\pi} P_0(x)$ results from the uniform distribution of the phase and in this way $P_0(x)$ acquires the physical meaning of the normalized distribution of the reflection parameter x.

Relationship (28) is one of our central results, which despite its apparent simplicity will have many far-reaching consequences which are not easy to guess otherwise. Let us note the invariance property of this distribution under the change $\mathrm{i}K \to 1/\mathrm{i}K$, meaning that both the impedance and its inverse (i.e. the admittance) must have one and the same probability distribution. Another remarkable feature of (28) is that it relates the distribution of the local Green function in the completely *closed* system to the distribution of the reflection coefficient in the perfectly *open* one. This fact will be fully utilized later on for extracting statistical properties of the Wigner time delay.

It is instructive to discuss first the *exact* limiting statistics

$$P_0(x) = \begin{cases} \dfrac{\alpha^{\beta/2+1}}{2\Gamma(\beta/2 + 1)} \left(\dfrac{x + 1}{2} \right)^{\beta/2} \mathrm{e}^{-\alpha(x+1)/2} & (\alpha \ll 1) \\[4mm] \dfrac{\alpha}{2}\, \mathrm{e}^{-\alpha(x-1)/2} & (\alpha \gg 1), \end{cases} \qquad \alpha \equiv \frac{1}{2}\beta\gamma, \tag{30}$$

which can be found [21] in the weak or strong absorption limit. In the former case, one can follow [15, 61] in using a perturbation theory to relate reflection to the time delay in the ideal cavity without absorption (see (6) thereafter). The latter quantity has a known distribution [40, 83] $\mathcal{P}_\tau\left(\tau = \frac{q}{t_H}\right) = \left[\left(\frac{\beta}{2}\right)^{\beta/2}/\Gamma\left(\frac{\beta}{2}\right)\right]\tau^{-\beta/2-2}\, \mathrm{e}^{-\beta/2\tau}$, which yields the first line above. In the opposite case of strong absorption, the real and imaginary parts of S-matrix become statistically independent Gaussian-distributed variables [14, 80], resulting in the so-called Rayleigh distribution $P(r) \simeq (\gamma\beta/2)\, \mathrm{e}^{-r\gamma\beta/2}$ of reflection valid at $r \approx \frac{1}{2}(x - 1) \ll 1$, and the second line in (30) follows.

As to arbitrary absorption, an exact result was obtained first for unitary symmetry ($\beta = 2$) by Beenakker and Brouwer [15]. It is convenient to keep the scaled absorption parameter α, representing their result in the following form:

$$P_0(x) = \frac{\mathcal{N}_\beta}{2} [A(\alpha(x+1)/2)^{\beta/2} + B] e^{-\alpha(x+1)/2} \tag{31}$$

with α-dependent constants $A \equiv e^\alpha - 1$ and $B \equiv 1 + \alpha - e^\alpha$, and with $\mathcal{N}_2 = 1$ standing for the normalization. For the case of symplectic symmetry ($\beta = 4$), the exact form was reported very recently [21]. The explicit derivation is given in appendix A, the final result being

$$P_0^{\text{gse}}(x) = \widetilde{P}_0^{\text{gue}}(x) + \left[\frac{1}{2}\gamma^2(x+1)^2 - \gamma(\gamma+1)(x+1) + \gamma \right] e^{-\gamma x} \int_0^\gamma dt \frac{\sinh t}{t}, \tag{32}$$

where $\widetilde{P}_0^{\text{gue}}(x)$ is the distribution (31) for $\beta = 2$ taken, however, at $\alpha = 2\gamma$. The case of orthogonal symmetry, the most welcomed experimentally, turns out to be extremely tricky. It was suggested in [21], see also [10], that expression (31) (with $\mathcal{N}_\beta = \alpha/(A\Gamma(\beta/2+1, \alpha) + B e^{-\alpha})$ and $\Gamma(\nu, \alpha) = \int_\alpha^\infty dp \, p^{\nu-1} e^{-p}$) is an appropriate interpolation formula at $\beta = 1$. It incorporates correctly both known limiting cases of weak or strong absorption and a reasonable agreement with available numerical and experimental data was found there in a broad range of the absorption strength. Below we provide an exact analytical treatment of this case by the method suggested in [22].

4.1.2. $\mathcal{P}(u, v)$ and the spectral correlation function of Green function resolvents. We establish now the general relation [22] between the joint distribution function (28) at arbitrary *finite* absorption and the energy autocorrelation function

$$C_\Omega(z_-, z_+) = \left\langle \frac{1}{z_- - i0 - K_0\left(E - \frac{1}{2}\Omega - i0\right)} \frac{1}{z_+ + i0 - K_0\left(E + \frac{1}{2}\Omega + i0\right)} \right\rangle \tag{33}$$

of the resolvents of the local Green function K_0 at zero absorption ($\Gamma = 0$, indicated explicitly in this subsection with the subscript '0'). Distribution (28) can be obtained from (33) by analytic continuation in Ω from a real to purely imaginary value $\Omega = i\Gamma$ as follows. $K_0(E)$ is an analytic function of the energy in the upper or lower half-plane and, thus, can be analytically continued to the complex values: $K_0\left(E \pm \frac{i}{2}\Gamma\right) \equiv u \mp iv, v > 0$. This allows us to continue then analytically the correlation function (33) from a pair of its real arguments to the complex conjugate one: $z_+ = (z_-)^* \equiv z' + iz'', z'' > 0$. As a result, function (33) acquires at $\Omega = i\Gamma$ the following form:

$$C(z', z'') \equiv C_{\Omega=i\Gamma}(z_-, z_+) = \left\langle \frac{1}{(z'-u)^2 + (z''+v)^2} \right\rangle$$

$$= \int_{-\infty}^\infty du \int_0^\infty dv \frac{\mathcal{P}(u, v)}{(z'-u)^2 + (z''+v)^2}. \tag{34}$$

The second line here is due to the definition (28). To solve this equation for $\mathcal{P}(u, v)$, we perform first the Fourier transform (FT) with respect to z' that leads to

$$\widehat{C}(k, z'') \equiv \int_{-\infty}^\infty dz' \, e^{ikz'} C(z', z'') = \int_0^\infty dv \, \widehat{\mathcal{P}}(k, v) \frac{\pi \, e^{-|k|(z''+v)}}{z'' + v}, \tag{35}$$

where $\widehat{\mathcal{P}}(k, v)$ is the corresponding FT of $\mathcal{P}(u, v)$. Being derived at $z'' > 0$, equation (35) can be analytically continued to the whole complex z'' plane with a cut along negative $\text{Re } z''$. Calculating then the jump of $\widehat{C}(k, z'')$ on the discontinuity line $z'' = -v$ ($v > 0$), we finally get the following expression:

$$\widehat{\mathcal{P}}(k, v) = \frac{\Theta(v)}{2\pi^2 i}[\widehat{C}(k, -v - i0) - \widehat{C}(k, -v + i0)] \qquad (36)$$

with the Heaviside step function $\Theta(v)$. The inverse FT of (36) yields $\mathcal{P}(u, v)$.

This completely general relationship is another our central result. It resembles (and reduces to) the well-known relation between the spectral density of states and the imaginary part of the corresponding resolvent operator, when the case of one real variable is considered. In contrast, the case of the distribution of two real variables requires to deal with the two-point correlation function. Physically, the latter is a generalized susceptibility describing a response of the system under consideration. This fact suggests to consider our formula in a sense as a 'fluctuation dissipation' relation: The lhs of (36) stands for the *distribution* (of K) in the presence of dissipation/absorption whereas the *correlation function* (of resolvents of K) in the rhs accounts for fluctuations in the system, i.e. for arbitrary order correlations in the absence of absorption.

The main advantage of the derived relation is that the correlation function is a much easier object to calculate analytically as compared to the distribution. Such a calculation for ideal systems at zero absorption has actually already been performed in many interesting cases. In the particular case of a chaotic cavity an exact result for the correlation function (33) has been previously derived by us in [40, 84]. Moreover, similar formulae can be derived for any model describing a one-particle quantum motion in a d-dimensional sample with a static disordered potential, see outline of the derivation in appendix B. The only important physical assumption is that the disorder is such that all the relevant statistical properties of the system can be adequately described by the standard diffusive supersymmetric nonlinear σ-model (or its lattice version). For a detailed discussion of the validity of this approximation (and its limitations) the interested reader is referred to the review [29] and the book [74].

In the 'zero-dimensional' case (chaotic cavity) the analytic continuation of (33) to complex $\Omega = i\Gamma$ can be represented generally as follows (see appendix B for details):

$$C(z', z'') = \frac{1}{z'^2 + (z'' + 1)^2} + \frac{1}{4}\left(\frac{\partial^2}{\partial z'^2} + \frac{\partial^2}{\partial z''^2}\right)\mathcal{F}(\tilde{x}), \qquad \tilde{x} \equiv \frac{z'^2 + z''^2 + 1}{2z''} \qquad (37)$$

where it is important that the function $\mathcal{F}(\tilde{x})$ depends on z' and z'' only via the scaling variable $\tilde{x} > 1$. Its explicit form depends on the symmetry present (e.g. preserved or broken TRS), the following common structure being however generic:

$$\mathcal{F}(\tilde{x}) = \int_{-1}^{1} d\lambda_0 \int_{1}^{\infty} d\lambda_1 \int_{1}^{\infty} d\lambda_2 \frac{f(\{\lambda\}) e^{-\gamma(\lambda_1\lambda_2 - \lambda_0)/2}(\tilde{x} + \lambda_0)}{\left[(\tilde{x} + \lambda_1\lambda_2)^2 - (\lambda_1^2 - 1)(\lambda_2^2 - 1)\right]^{1/2}}. \qquad (38)$$

The *real* function $f(\{\lambda\})$ is the only symmetry dependent term here. In the crossover regime of gradually broken TRS it can be represented explicitly as follows [84]:

$$f(\{\lambda\}) = \left\{(1 - \lambda_0^2)(1 + e^{-2Y}) - (\lambda_1^2 - \lambda_2^2)(1 - e^{-2Y})\right.$$
$$\left. + 4y^2 \mathcal{R}\left[(1 - \lambda_0^2) e^{-2Y} + \lambda_2^2(1 - e^{-2Y})\right]\right\}\frac{e^{-2y^2(\lambda_2^2 - 1)}}{\mathcal{R}^2} \qquad (39)$$

with $\mathcal{R} = \lambda_0^2 + \lambda_1^2 + \lambda_2^2 - 2\lambda_0\lambda_1\lambda_2 - 1$ and $Y \equiv y^2(1 - \lambda_0^2)$, where y denotes a crossover driving parameter. Physically, $y^2 \sim \delta E_y/\Delta$ is determined by the energy shift δE_y of energy levels due to a TRS breaking perturbation (e.g., weak external magnetic field in the case of quantum dots) [85, 86]. Such an effect is conventionally modelled within the framework of RMT by means of the 'Pandey–Mehta' Hamiltonian [87], $H = \hat{H}_S + i(y/\sqrt{N})\hat{H}_A$, with $\hat{H}_S$ ($\hat{H}_A$) being a random real symmetric (antisymmetric) matrix with independent Gaussian-distributed entries. The limit $y \to 0$ or $y \to \infty$ corresponds to fully preserved or broken TRS, respectively.

Now we apply relation (36) to equation (37) and then perform the inverse FT to get $\mathcal{P}(u, v)$. Relegating all technical details to appendix C, we emphasize here the most important points. The nontrivial contribution to the distribution comes from the second ('connected') part of the correlation function (37) whereas the first ('disconnected') one is easily found to yield the singular contribution $\delta(u)\delta(v-1)$. A careful analysis shows that due to specific $\tilde{x}$-dependence given by equation (38) the above-described procedure for the analytic continuation of the connected part of $\widehat{C}(k, z'')$ is equivalent to continuing $\mathcal{F}(\tilde{x})$ analytically and taking the jump at

$$\tilde{x} = -x \pm \mathrm{i}0, \qquad \text{with} \qquad x \equiv \frac{1}{2v}(u^2 + v^2 + 1) > 1. \qquad (40)$$

Thus, the nonzero imaginary part $F(x) = \operatorname{Im} \mathcal{F}(-x + \mathrm{i}0)$ of the analytic continuation of (38) is determined at given x by the following integral:

$$F(x) = \int_{-1}^{1} \mathrm{d}\lambda_0 \iint_{\mathcal{B}_x} \mathrm{d}\lambda_1 \, \mathrm{d}\lambda_2 \frac{f(\{\lambda\}) \, \mathrm{e}^{-\gamma(\lambda_1\lambda_2 - \lambda_0)/2}(x - \lambda_0)}{\left[(\lambda_1^2 - 1)(\lambda_2^2 - 1) - (\lambda_1\lambda_2 - x)^2\right]^{1/2}} \qquad (41)$$

over the integration region $\mathcal{B}_x = \left\{(\lambda_1, \lambda_2) | 1 \leqslant \lambda_{1,2} < \infty, (\lambda_1\lambda_2 - x)^2 < (\lambda_1^2 - 1)(\lambda_2^2 - 1)\right\}$, where the square root in (38) attains pure imaginary values. Taking now into account the identity $\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)F(x) = v^{-2}\frac{\mathrm{d}}{\mathrm{d}x}(x^2 - 1)\frac{\mathrm{d}}{\mathrm{d}x}F(x)$, which is valid for $x^2 \neq 1$, we finally arrive at

$$\mathcal{P}(u, v) = \frac{1}{4\pi^2 v^2} \frac{\mathrm{d}}{\mathrm{d}x}(x^2 - 1)\frac{\mathrm{d}F(x)}{\mathrm{d}x} \equiv \frac{1}{2\pi v^2} P_0(x). \qquad (42)$$

This proves in general, cf (28), the statistical independence and uniform distribution of the scattering phase at perfect coupling. At arbitrary values of the crossover parameter y the obtained expression can be further treated only numerically. However, further analytical progress is possible in the limiting cases of pure symmetries and is considered below. We discuss explicitly the two most important cases of unitary and orthogonal symmetries. Clearly, the general scheme can be extended to the symplectic case with no principal difficulties.

4.1.3. Unitary symmetry. This case corresponds to considering the limit $y \to \infty$. The nonzero contribution comes then from the second line in (39) where one can use effectively $4y^2 \, \mathrm{e}^{-2y^2(\lambda_2^2 - 1)} \to \delta(\lambda_2 - 1)$ in the limit under discussion. This simplifies expression (38) further to $\mathcal{F}(\tilde{x}) = \int_1^{\infty} \mathrm{d}\lambda_1 \int_{-1}^{1} \frac{\mathrm{d}\lambda_0}{(\lambda_1 - \lambda_0)^2} \frac{\tilde{x} + \lambda_0}{\tilde{x} + \lambda_1} \mathrm{e}^{-\gamma(\lambda_1 - \lambda_0)/2}$. Performing now analytical continuation (40) and making use of $\operatorname{Im} \frac{1}{\lambda_1 - x + \mathrm{i}0} = -\pi\delta(\lambda_1 - x)$, one gets readily $F(x) = \pi \int_{-1}^{1} \mathrm{d}\lambda_0 (x - \lambda_0)^{-1} \mathrm{e}^{-\gamma(x - \lambda_0)/2}$, which yields the distribution [17]

$$P_0^{\mathrm{gue}}(x) = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}x}(x^2 - 1)\frac{\mathrm{d}}{\mathrm{d}x} \int_{x-1}^{x+1} \frac{\mathrm{d}t}{t} \, \mathrm{e}^{-\gamma t/2} \qquad (43)$$

leading to equation (31) with $\beta = 2$ obtained earlier [15] by a different method.

4.1.4. Orthogonal symmetry. This case amounts to investigating (39) and (41) at $y = 0$. Fortunately, further simplifications are possible if one considers the integrated probability distribution $W(x) \equiv -\frac{1}{2\pi}(x^2 - 1)\frac{\mathrm{d}}{\mathrm{d}x}F(x) = \int_x^{\infty} \mathrm{d}x \, P_0(x)$, which is a positive monotonically decaying function by definition. To this end, we note that it is useful to switch to the parametrization of [2] to carry out the three-fold integration. The latter turns out to yield then a sum of decoupled terms and, after some algebra (see appendix C), the result can be cast [22] in the following final form:

$$W(x) = \frac{x+1}{4\pi}[f_1(w)g_2(w) + f_2(w)g_1(w) + h_1(w)j_2(w) + h_2(w)j_1(w)]_{w=\frac{x-1}{2}} \qquad (44)$$

with auxiliary functions defined as follows:

$$f_1(w) = \int_w^\infty dt \frac{\sqrt{t|t-w|}\, e^{-\gamma t/2}}{(1+t)^{3/2}} [1 - e^{-\gamma} + t^{-1}],$$

$$g_1(w) = \int_w^\infty dt \frac{1}{\sqrt{t|t-w|}} \frac{e^{-\gamma t/2}}{(1+t)^{3/2}},$$

$$h_1(w) = \int_w^\infty dt \frac{\sqrt{|t-w|}\, e^{-\gamma t/2}}{\sqrt{t(1+t)}} [\gamma + (1 - e^{-\gamma})(\gamma t - 2)],$$

$$j_1(w) = \int_w^\infty dt \frac{1}{\sqrt{t|t-w|}} \frac{e^{-\gamma t/2}}{\sqrt{1+t}};$$

their counterpart with the index 2 being given by the same expression save for the integration region $t \in [0, w]$ instead of $[w, \infty)$. The interpolation expression (31) with $\beta = 1$ compared to the above exact result shows systematic deviations in the nonperturbative regime of moderate absorption $\gamma \sim 1$ (see [22]). Both formulae essentially coincide in the limiting cases of weak or strong absorption, when one can already use the more simple and physically transparent exact limiting statistics (30).

4.1.5. Impedance, reactance and the LDoS. Now we discuss statistics of the real and imaginary parts of the local Green function one by one. Let us consider first the distribution of the imaginary part v (LDoS). Having $P_0(x)$ at our disposal, it is immediate to find the distribution function $\mathcal{P}_v(v)$ of v by integrating out u in (28):

$$\mathcal{P}_v(v) = \frac{\sqrt{2}}{\pi v^{3/2}} \int_0^\infty dq\, P_0 \left[q^2 + \frac{1}{2}\left(v + \frac{1}{v} \right) \right]. \tag{45}$$

This distribution is normalized to 1 and has the first moment unity at arbitrary absorption (indeed $\langle v \rangle \equiv \int_0^\infty dv\, v\mathcal{P}(v) = 1$ is automatically satisfied due to invariance of the integrand of (45) with respect to the change $v \to \frac{1}{v}$). Explicit results for $\mathcal{P}_v(v)$ as well as for $\mathcal{P}_u(u)$ (see (48)) can be readily obtained as $P_0(x)$ at arbitrary γ has been already derived above. We refer the reader to the original publication [21], focusing now on the physically interesting limiting cases of weak and strong absorption, where the behaviour of the distributions is qualitatively different.

At $\gamma \ll 1$, the exact limiting statistics (30) for $P_0(x)$ can be used to perform the integration in (45). One arrives at the following result:

$$\mathcal{P}_v(v) \propto \begin{cases} \alpha^{(1+\beta)/2} v^{-(3+\beta)/2}\, e^{-\alpha/4v}, & v \ll \alpha \\ \alpha^{1/2} v^{-3/2}, & \alpha \ll v \ll 1/\alpha \\ \alpha^{(1+\beta)/2} v^{-(3-\beta)/2}\, e^{-\alpha v/4}, & 1/\alpha \ll v, \end{cases} \tag{46}$$

where constant factors of the order unity are omitted. This result can be physically interpreted in the single-level approximation [88, 89], when the main contribution to the LDoS $v \approx v_n = |\psi_n|^2 \frac{\gamma}{2\pi}(\varepsilon_n^2 + \frac{1}{4}\gamma^2)^{-1}$ comes from a level closest to the given value of the energy ($\varepsilon_n = 2\pi(E - E_n)/\Delta$). Fluctuations of the wavefunction ψ_n are mainly responsible for the exponential suppression of the distribution on the tails, while spectral fluctuations produce the intermediate bulk behaviour with a 'super-universal' $\frac{3}{2}$-power law, which actually does not depend on the symmetry at all. As γ increases, a number of levels contributing to $v \sim \sum v_n$ also grow $\sim\gamma$ and their contributions get less correlated. The resulting limiting form is almost a Gaussian:
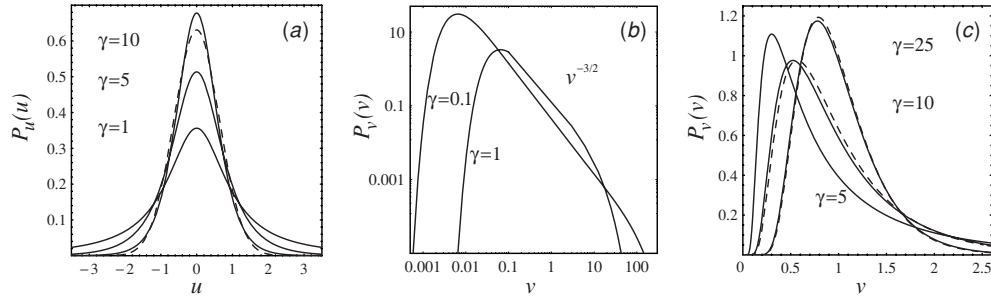
Figure 2. The distribution of the real u and imaginary v parts of the local Green function in the time-reversal chaotic cavity as absorption γ grows. Dashed lines in (a) and (c) correspond to the limiting statistics (49) and (47), respectively.

$$\mathcal{P}_v(v) = \sqrt{\frac{\alpha}{4\pi v^3}} \exp\left[-\frac{\alpha}{4}\left(\sqrt{v} - \frac{1}{\sqrt{v}}\right)^2\right], \qquad \alpha \gg 1. \tag{47}$$

It shows a peak at $v \sim 1$ of the width $\propto 1/\sqrt{\gamma} \ll 1$, in agreement with [89].

Along the same lines, we may consider the distribution $\mathcal{P}_u(u)$ of the real part u ('reactance'). One finds from (28)

$$\mathcal{P}_u(u) = \frac{1}{2\pi\sqrt{u^2+1}} \int_0^\infty \mathrm{d}q\, P_0\left[\frac{\sqrt{u^2+1}}{2}\left(q + \frac{1}{q}\right)\right]. \tag{48}$$

The limiting forms of $\mathcal{P}_u(u)$ at weak and strong absorption follow readily as

$$\mathcal{P}_u(u) \simeq \begin{cases} \pi^{-1}(1+u^2)^{-1}, & \alpha \ll 1 \\ \sqrt{\alpha/4\pi}\, \mathrm{e}^{-\alpha u^2/4}, & \alpha \gg 1. \end{cases} \tag{49}$$

The Lorenzian distribution in the first line of (49) is the consequence of the uniform distribution of the scattering phase [90], as $u \approx \cot\frac{\theta}{2}$ in the limit considered. As absorption grows, one can see the crossover to a Gaussian distribution centred at zero, which is again a result of the central limit theorem. This type of behaviour as well as a trend of $\mathcal{P}_v(v)$ to the Gaussian distribution (47), see figure 2, was recently observed in experiments on the cavity impedance [25, 26].

4.1.6. Delay time at vanishing absorption and eigenfunction intensity. The Wigner time delay is one of the most important and frequently used characteristics of quantum scattering. It was recently realized [35] that the fundamental relation (28) allows one to relate the distribution of the time delay q in single-channel scattering (i.e. the Wigner time delay) from a general *lossless* disordered system to the distribution of wavefunction intensities in the closed counterpart of such a system. Namely, consider the local eigenfunction intensity $y = \mathcal{V}|\psi_n(\mathbf{r})|^2$ at a spatial point $\mathbf{r}$ of the closed system, with $\mathcal{V}$ denoting the volume of the sample and the index n numbering different eigenfunctions. Let $\mathcal{P}_y(y)$ stand for the distribution of this intensity, where the statistics is sampled both over various realizations of the disorder in the system and over a certain small energy range around the point E in the spectrum, with Δ being the mean level spacing in that energy range. Define the dimensionless time delay $\tau_w = q\Delta/2\pi$ corresponding to wave scattering at the chosen energy E from a single channel attached to the point $\mathbf{r}$ of the sample. Denoting by $\mathcal{P}_w(\tau_w)$ the corresponding distribution in the regime of perfect coupling to the sample, Ossipov and Fyodorov managed to derive the following

functional relation between the two distributions:

$$\mathcal{P}_w(\tau_w) = \tau_w^{-3} \mathcal{P}_y(\tau_w^{-1}). \tag{50}$$

Referring the interested reader for the details of derivation to [35], we would like only to mention that at the starting point of the derivation the time delay is expressed via the reflection coefficient r in presence of absorption: $\tau_w = \lim_{\gamma \to 0} \frac{1-r}{\gamma}$, see (6). Then one exploits in a clever way the scaling form of the distribution (28) remembering both the interpretation of the variable x in terms of r, see (29), and the interpretation of v as the LDoS, the latter step providing a connection to eigenfunction statistics.

On one hand, in the idealized situation of zero absorption statistics of delay times of all sorts were studied intensively in the framework of the RMT approach, various exact analytical results being available, see [40, 49, 83, 84, 91–96] and references therein. Those were successfully verified in numerical simulations of chaotic systems of quite a diverse nature, see [97–99]. Since phase shifts and delay times are experimentally measurable quantities, especially in a single-channel reflection experiment [5, 8, 10, 100–103], relation (50) opens a new possibility for experimental study of eigenfunctions.

On the other hand, one can use the existent knowledge on eigenfunction statistics [29, 104] to provide via (50) explicit expressions for time-delay distributions. In this way one can e.g. recover those for chaotic systems of all symmetry classes obtained previously by diverse methods in various regimes of interest [40, 83, 84]. Of particular interest is the predicted multifractal behaviour of the negative moments of time delays in the vicinity of the Anderson localization transition [35]:

$$\langle \tau_w^{-q} \rangle \propto L^{-q D_{q+1}}, \tag{51}$$

where L stands for the system size at criticality, and D_q are anomalous (multifractal) dimensions of the eigenfunctions. Such a behaviour was indeed discovered recently [105] in numerical simulations of the disordered lattice Hamiltonians at criticality.

4.1.7. Anderson transition as phenomenon of spontaneous breakdown of S-matrix unitarity. Actually, absorption in disordered systems plays not only purely technical, but also a conceptually important role in revealing the mechanisms behind the Anderson localization transition. Let us shortly discuss a possible qualitative behaviour of the PNR $\tau = 1 - r$ in a scattering system formed by a single perfect channel attached to a d-dimensional disordered sample at the vicinity of the point of the Anderson delocalization transition μ_c (the mobility edge). Here we denote by μ an effective parameter which controls the transition in the infinite sample, with states being localized (extended) for $\mu > \mu_c$ (respectively, $\mu < \mu_c$).

For a sample of finite size L the PNR is a function of three parameters: absorption Γ, size L and disorder strength μ. In the insulating phase $\mu > \mu_c$ the system is characterized by a localization length ξ which diverges when μ approaches the critical value. A natural scale for the absorption is played by a parameter $\delta \propto \Gamma \xi^d$, i.e. the ratio of the imaginary shift in energy to the mean level spacing for sample of a typical size $L \propto \xi$ (localization volume).

Consider the weak absorption limit $\delta \to 0$. It is clear that the wave incoming through the incident channel effectively explores only a part of the sample of the order of localization volume ξ^d, being exponentially small elsewhere. Under this condition it is clear that the two limits $\Gamma \to 0$ and $L \to \infty$ should actually commute and can be taken in arbitrary order. We know that in the limit $\Gamma \to 0$ the PNR behaves as $\tau \approx \Gamma q \to 0$, see (6). Moreover, exploiting relation (50), and remembering that $\langle |\psi_n(\mathbf{r})|^{2k} \rangle \propto \xi^{-(k-1)d}$ it is easy to see that all negative integer PNR moments should behave in the infinite volume limit $L \to \infty$ as $\langle \tau^{-k} \rangle \sim \Gamma^{-k} \xi^{-dk}$. With a little more work one can suggest a qualitative picture for the

PNR probability distribution in the localized phase. Namely, the distribution should have a power-law tail $\mathcal{P}(\tau) \propto \delta/\tau^2$ extending through a parametrically large domain $\delta < \tau < 1$, and should decay very fast towards zero for both $\tau \ll \delta$ and $\tau \to 1$. When absorption vanishes $\delta \propto \Gamma \to 0$ such a distribution collapses to the Dirac δ-function $\mathcal{P}(\tau) \propto \delta(\tau)$, but in a very nonuniform way. We may conclude that in the limit of vanishing absorption S-matrix unitarity is indeed recovered, and in this sense we can associate the localized phase with the phase of unbroken symmetry.

In contrast, in the delocalized phase the incoming wave explores the whole sample volume. It is natural to think that whatever small (but fixed) is an absorption rate Γ, in the limit $L \to \infty$ a finite portion of the incoming flux will be absorbed in the sample and will never come back to the incident channel. In particular, we may expect $\lim_{\Gamma \to 0} \lim_{L \to \infty} \tau(\Gamma, L, \mu) = \tau_\infty(\mu) > 0$ as long as $\mu < \mu_c$. From this point of view the Anderson transition acquires a natural interpretation as the phenomenon of *spontaneous* breakdown of S-matrix unitarity. Informal arguments in favour of such a behaviour are based on a picture of the transition as described in terms of a functional order parameter developed in detail in [81, 82], see also earlier results in [74] and [106]. Namely, the distribution function $\mathcal{P}(\tau)$ is expected to remain a nontrivial finite-width distribution even when $\Gamma \to 0$, provided the latter limit is taken *after* the infinite volume limit $L \to \infty$. Clearly, more work is needed to substantiate this claim, as well as to clarify critical behaviour of $\tau_\infty(\mu)$ as long as $\mu \to \mu_c$.

Note finally that if the limit $\Gamma \to 0$ is taken first, then for $\mu < \mu_c$ PNR in large but finite sample should again scale with the system size L as $\langle \tau^{-k} \rangle \sim C(\mu) \Gamma^{-k} L^{-dk}$, cf (51). The coefficient $C(\mu)$ is however expected to diverge when $\mu \to \mu_c$. This divergence should be related to the properties of eigenfunctions via equation (50).

4.1.8. Arbitrary coupling to the channel. The general case of arbitrary transmission coefficient, $T < 1$, can be mapped [10, 95, 107, 108] to that of perfect coupling by making use of the following relation:

$$S_{T=1} = \frac{S - \sqrt{1-T}}{1 - \sqrt{1-T}\,S} \tag{52}$$

between the corresponding scattering matrices. Equation (52) is known from the Poisson kernel theory [107]. Due to an additional interference between incoming and directly back-scattered waves, the scattering phase θ acquires a nonuniform distribution and statistical correlations with x (or r). However, the joint probability density $P(x, \theta)$ is again determined by the function P_0 as follows [21]:

$$P(x, \theta) = \frac{1}{2\pi} P_0\big(xg - \sqrt{(x^2-1)(g^2-1)}\cos\theta\big), \qquad g = \frac{2}{T} - 1 \geqslant 1. \tag{53}$$

This relation can be obtained by a straightforward evaluation in the parametrization (52) of the corresponding Jacobian. The integration over x immediately yields the scattering phase distribution. In the particular case of vanishing absorption $r \to 1$ and $P_0(x) \to x^{-2}\delta(1/x)$. This readily gives $P(x, \theta) \to \rho(\theta)P_0(x)$, with the phase density $2\pi\rho(\theta) = \big(g - \sqrt{g^2-1}\cos\theta\big)^{-1}$ found earlier [95] (see [99] for the corresponding numerical study). As another example we consider the reflection coefficient distribution in terms of $P_0(x)$ is given at arbitrary T by [17]

$$P_r(r) = \frac{1}{\pi(1-r)^2} \int_0^{2\pi} \mathrm{d}\theta\, P_0\left[\frac{2\big(g - \sqrt{g^2-1}\sqrt{r}\cos\theta\big)}{1-r} - g\right]. \tag{54}$$

Distributions of r and θ in microwave cavities were recently studied for different realizations of γ and T in [10], the excellent agreement with the theory being found.

4.2. Beyond single channel

4.2.1. Reflection coefficients and PNRs.
The starting point of our analysis in this section is the following convenient representation [17] for the diagonal elements $S^{cc}(E)$ of the scattering matrix, cf (1):

$$S^{cc}(E) = \frac{1 - iK_c(E)}{1 + iK_c(E)}, \qquad K_c(E) = \frac{1}{2} V^{c\dagger}\left(E - \mathcal{H}^c_{\text{eff}}\right)^{-1} V^c, \qquad (55)$$

where $\mathcal{H}^c_{\text{eff}} = H - \frac{i}{2}\sum_{b\neq c} V^b V^{b\dagger}$ is now the c-dependent non-Hermitian operator. In view of $\mathcal{H}_{\text{eff}} = H - \frac{i}{2} V V^{\dagger} \equiv \mathcal{H}^c_{\text{eff}} - \frac{i}{2} V^c V^{c\dagger}$ one can treat $V^c V^{c\dagger}$ as a rank 1 perturbation with respect to $\mathcal{H}^c_{\text{eff}}$. In this case the following general relationship (Dyson's equation) is valid for the corresponding resolvents [38]:

$$\frac{1}{E - \mathcal{H}_{\text{eff}}} = \frac{1}{E - \mathcal{H}^c_{\text{eff}}} - \frac{i}{2}\frac{1}{E - \mathcal{H}^c_{\text{eff}}} V^c \frac{1}{1 + iK_c(E)} V^{c\dagger}\frac{1}{E - \mathcal{H}^c_{\text{eff}}} \qquad (56)$$

which can be proved by expanding the lhs in a power series with respect to $\left(E - \mathcal{H}^c_{\text{eff}}\right)^{-1} V^c V^{c\dagger}$ and summing then up the resulting geometric series in K_c. Substituting this equation in $S^{cc} = 1 - iV^{c\dagger}(E - \mathcal{H}_{\text{eff}})^{-1} V^c$ one gets $S^{cc} = 1 - 2iK_c - 2K_c(1 + iK_c)^{-1} K_c$ which is equivalent to (55). It is also worth noting that a representation like (55) is valid for an arbitrary $m \times m$ submatrix standing on the main diagonal of S with obvious replacement V^c by the $N \times m$ matrix $V^{(m)} = (V^{c_1}, \ldots, V^{c_m})$, and corresponding changes for K_c and $\mathcal{H}^c_{\text{eff}}$. In the particular case $m = M$, i.e. the full S-matrix, one recovers (1) from (2) and (56).

Relation (55) reduces the problem of evaluating the statistics of S_{cc}, and hence the reflection coefficient/PNR in a given channel c to calculate the joint probability density $\mathcal{P}(u_c, v_c)$ of $u_c = \text{Re}\, K_c$ and $v_c = -\text{Im}\, K_c$, with K_c standing for the particular diagonal entry of the resolvent of the effective *non-Hermitian* Hamiltonian $\mathcal{H}^c_{\text{eff}}$ from (55). Moreover, a uniform absorption within the sample can again be taken into account by a purely imaginary shift of the scattering energy $E \to E + \frac{i}{2}\Gamma \equiv E_\gamma$, all further steps being fully in parallel to those of section 4.1.2. The most pleasant feature of this approach is that it is very straightforward to include open channels in the derivation of $\mathcal{P}(u_c, v_c)$ within the supersymmetry method. All important properties of this distribution, in particular, relations (28), (41) and (42) retain its validity, with x being naturally replaced by $x_c = \frac{1}{2v_c}\left(u_c^2 + v_c^2 + 1\right)$. The corresponding function $F(x_c)$ then follows from (41) by multiplying there the integrand with the 'channel factor' (which originates from the imaginary part of $\mathcal{H}^c_{\text{eff}}$)

$$\mathcal{F}^c_M(\{\lambda\}) = \prod_{b\neq c} \left[\frac{(g_b + \lambda_0)^2}{(g_b + \lambda_1\lambda_2) - \left(\lambda_1^2 - 1\right)\left(\lambda_2^2 - 1\right)}\right]^{1/2}, \qquad g_b = \frac{2}{T_b} - 1, \qquad (57)$$

which accounts for *arbitrary* coupling to all other channels save for the given perfectly open one c. Arbitrary coupling for the remaining channel c can be considered by means of (53), providing us finally with the general distribution of the reflection amplitude and phase. In this way one can also obtain explicit distributions for many interesting situations [17], including the cases when the effects of Anderson localization start to play an important role. In particular, in the case of broken TRS one can calculate the PNR distribution for a single channel attached to an edge of a piece of quasi-one-dimensional disordered medium of a given length L, with the opposite edge being either closed or in contact with a perfectly conducting multichannel lead.

4.2.2. Reflection eigenvalues and thermal emission.
The exact result for the distribution function $P(r)$ of reflection eigenvalues valid for any number of arbitrary open channels and

arbitrary absorption is known only for the $\beta = 2$ case, being recently calculated by two of us [16], generalizing earlier perturbative results [15] (known for all β). That uses the method developed for studying the proper time-delay distribution [96] in the ideal lossless system. The later distribution considered at finite absorption by means of (6) has a sharp cutoff at $q = \Gamma^{-1}$ due to a finite value of $P(r)$ at $r = 0$. This fact makes the interpretation of q as delay times at strong absorption questionable [16], since intuitively one expects a generic exponential suppression at large values of delay times $q \gg \Gamma^{-1}$. Indeed, for the time δt a wave-packet oscillating in the cavity with a frequency $\Delta/2\pi$ on average experiences $(\Delta/2\pi)\delta t$ collisions with the walls, yielding the probability $T_\phi(\Delta/2\pi)\delta t$ to be absorbed into one of M_ϕ parasitic channels ($T_\phi \ll 1$ being the transmission coefficient). The total reflection is then estimated as $R \simeq (1 - T_\phi(\Delta/2\pi)\delta t)^{M_\phi}$, giving $\mathrm{e}^{-\Gamma\delta t}$ in the absorption limit of the fixed absorption width $\Gamma = M_\phi T_\phi \Delta/2\pi$ as $M_\phi \to \infty$ and $T_\phi \to 0$. It is natural, therefore, to define alternatively the positive definite matrix Q_R of *reflection* time delays as follows [16]:

$$Q_R \equiv -\Gamma^{-1} \ln R_\gamma = -\Gamma^{-1} \ln(1 - \Gamma Q_\gamma). \tag{58}$$

One finds easily the connection $\mathcal{P}_R(q_r) = \mathrm{e}^{-\Gamma q_r} \mathcal{P}\left[\frac{1}{\Gamma}(1 - \mathrm{e}^{-\Gamma q_r})\right]$ between the corresponding distributions $\mathcal{P}_R(q_r)$ and $\mathcal{P}(q)$ of proper delay times (eigenvalues of Q_R and Q_γ, respectively). Both Q_R and Q_γ reduce to the same Wigner–Smith matrix (4) in the limit of vanishing absorption. The difference between them becomes noticeable at finite Γ. Still both distributions coincide up to the time $\ll \Gamma^{-1}$. They start to differ at larger times, when $\mathcal{P}(q)$ has the cutoff whereas $\mathcal{P}_R(q \gg \Gamma^{-1}) \propto \mathrm{e}^{-\Gamma q}$ is exponentially suppressed.

The exact result for the $\beta = 1$ case at arbitrary M is still outstanding. In the limit of the large number of equivalent channels, $M \to \infty$, an exact result can be found at arbitrary absorption and coupling [18], and the perfect coupling case $T = 1$ has been known for some time [109, 110]. In contrast to the few-channel case, when any value $0 \leqslant r \leqslant 1$ is permitted, the distribution density in the present case is non-vanishing only in a range $0 \leqslant r_{\min} \leqslant r \leqslant r_{\max} < 1$, and is the same for all β. Referring the reader to [18] for explicit results, we mention their application for thermal emission from random media. Registration of n photons in the frequency window $\delta\Omega$ during the large time $t \gg 1/\delta\Omega$ yields the negative-bimodal distribution of photocounts with $\nu = Mt\delta\Omega/2\pi$ degrees of freedom; see [111]. In his seminal paper Beenakker [109] has shown that the quantum optical problem of the photon statistics can be reduced to a computation of the S-matrix of the classical wave equation. In particular, chaotic radiation may be characterized by the effective number ν_{eff} degrees of freedom as follows [109]: $\nu_{\mathrm{eff}}/\nu = (1 - \langle r \rangle)^2/\langle (1 - r)^2 \rangle \leqslant 1$, with $\nu_{\mathrm{eff}} = \nu$ for blackbody radiation. This ratio is given by $\nu_{\mathrm{eff}}/\nu = (\gamma_s + T)^2/[\gamma_s^2 + 2(\gamma_s + T)]$ at arbitrary transmission T and absorption $\gamma_s \equiv 2\pi\Gamma/\Delta M$, implying thus that saturation to the blackbody limit slows down, $\frac{\nu_{\mathrm{eff}}}{\nu} \approx 1 - \frac{2}{\gamma_s}(1 - T)$ at $\gamma_s \gg 1$, for transmission $T < 1$ [18]. Finally, due to a duality relation [109, 112] of a linear amplifying system to a dual absorbing one (related to it by inverting the sign of Γ) there exists a further link of the analysis presented here to the rapidly developing field of random lasers [109, 110, 112–114].

4.2.3. Off-diagonal entries of the Green function. As we have seen, the statistics of real and imaginary parts of diagonal elements of the Green function can be very efficiently studied in the framework of the supersymmetric approach and may have various physical interpretations. The off-diagonal entries $G_{ij}(E_\gamma) = \langle i|(E_\gamma - \mathcal{H}_{\mathrm{eff}})^{-1}|j\rangle$ are of considerable importance as well. It can be easily understood that $W = |G_{ij}|^2$ is essentially the wave power transmitted from a source at site i inside a random medium to a receiver at site j. Statistics of such an object is much more difficult to study in general. Presently the most studied case [19] is $\beta = 2$ symmetry class under an additional assumption that both receiver and source are very

weakly coupled to the medium as to ensure those couplings do not contribute essentially to the resonance broadening. This means that the broadening is induced purely due to losses elsewhere in the medium. Technically the latter requirement amounts to vanishing coupling amplitudes V_i^c and V_j^c for all damping channels $c = 1, \ldots, M$. Assuming further the RMT statistics for $\mathrm{Re}\,\mathcal{H}_{\mathrm{eff}}$, it is easy to see that the damping matrix $\mathrm{Im}\,\mathcal{H}_{\mathrm{eff}}$ can be chosen diagonal and simply such that both entries $(VV^\dagger)_{ii}$ and $(VV^\dagger)_{jj}$ are vanishing. For such a model the distribution of the scaled transmitted power $w = \left(\frac{\Delta}{\pi}\right)^2 W$ can be found explicitly [19] as

$$\mathcal{P}(w) = \left(\frac{\mathrm{d}}{\mathrm{d}w} + w\frac{\mathrm{d}^2}{\mathrm{d}w^2}\right)\left[\frac{\mathrm{e}^{-\gamma\sqrt{1+w}}}{2\sqrt{1+w}}\int_{-1}^{1}\mathrm{d}\lambda\frac{\sqrt{1+w}+\lambda}{\sqrt{1+w}-\lambda}\,\mathrm{e}^{\gamma\lambda}\prod_{c=1}^{M}\frac{g_c+\lambda}{g_c+\sqrt{1+w}}\right]. \tag{59}$$

In fact, the remaining integration can be performed for two interesting cases: (i) all equivalent dissipation channels $g_c = g$ in the absence of uniform losses $\gamma = 0$ and (ii) no internal channels of dissipation $g_c \to \infty$ in the presence of uniform absorption $\gamma > 0$. Here we present the formula only for the latter case [19]:

$$\mathcal{P}(w) = \frac{\gamma^4\,\mathrm{e}^{-\gamma_w}\sinh\gamma}{4\gamma_w^5}\left[\gamma_w^2(w+2) - \left(w - 2 - 2\left(\gamma_w^2/\gamma\right)\coth\gamma\right)(1+\gamma_w)\right] \tag{60}$$

with the shorthand $\gamma_w \equiv \gamma\sqrt{1+w}$. The distribution of the transmitted power for the case of absorptive media with preserved time-reversal invariance is not yet known. First two moments of that quantity were calculated recently in [20].

Let us finally mention that a closely related question of statistics of intensity of waves emitted from a permanently radiating source embedded in a random medium was the subject of many studies in recent years. Some useful references can be found in section 7 of [29], see also the relevant review [114] on random lasing.

5. Conclusions and open problems

For wave scattering in open chaotic and/or disordered systems with uniform losses, we have discussed various statistics on the level of both correlation and distribution functions. The overall exponential decay due to uniform absorption is the generic feature of any correlation function reduced to a two-point spectral (resolvent) correlation function, which follows simply from analytic properties of the latter in the complex energy plane. For fully chaotic systems with or without TRS, we have calculated exactly energy correlation functions of complex impedance and S-matrix elements at arbitrary absorption and coupling. The corresponding enhancement factors have been also discussed in detail. The result for S-matrix correlations in the case of broken TRS completes the well-known one [2] of preserved TRS.

To study distribution functions, we have described the novel approach to the problem by deriving a kind of a fluctuation dissipation relation (36) of quite a general nature, which relates distributions at finite absorption to arbitrary order correlations at zero absorption via a nontrivial analytic continuation procedure. Correlations can be efficiently studied in the framework of the supersymmetric nonlinear σ-model. In the zero-dimensional case, a number of explicit results is provided for quantities characterizing open chaotic systems both from 'inside' (LDoS, the complex impedance and the (local) Green function) and from 'outside' (PNRs, reflection coefficients and time delays). This σ-model mapping provides an attractive possibility to include into consideration open disordered absorptive systems beyond the usual RMT treatment.

Finally, let us briefly discuss an (incomplete) list of problems deserving, from our personal point of view, further investigations both for scattering systems with and without absorption.

Within the domain of RMT applicability, the most outstanding problems are of course those requiring evaluation of four-point correlation functions of the resolvents, e.g., partial scattering cross-sections. These are still not known even for the simplest case of broken TRS. Another challenge is to find a probability distribution for the multichannel *S*-matrix at finite absorption, thus nontrivially generalizing the Poisson kernel [90]. More work is required to understand an interplay between the statistics of complex resonances and the corresponding bi-orthogonal eigenfunctions [38, 69], the question being of particular relevance for lasing from random and/or chaotic media [115, 116]. Spatial characteristics of internal parts $b^c = (E - \mathcal{H}_{\text{eff}})^{-1} V^c$ of the scattering wavefunction are very interesting by their own and are closely related to fluctuations in transmitted power discussed above in section 4.2.3.

Apart from that, in the last decade it was realized [117] that it may be required to consider other symmetry classes (beyond the three of Dyson) which are relevant, e.g., for systems involving superconducting elements. The corresponding scattering theory is reviewed in [118]. To reconsider many problems discussed in the present paper taking those (and other related) symmetries into account should be both interesting and informative (see [24] as a recent example).

Last but not least, understanding scattering in disordered systems beyond the universal RMT regime, taking into account, in particular, the effects of Anderson localization, is a rather promising area calling for more systematic research.

Acknowledgments

Appendix A. Statistics of the local Green function in the GSE

We start with expressing the dimensionless LDoS (the imaginary part of the local Green function $G\left(E + \frac{\mathrm{i}}{2}\Gamma; \mathbf{r}; \mathbf{r}\right)$ in units of Δ) in terms of the eigenfunctions and eigenvalues of the Hamiltonian H with underlying simplectic symmetry as

$$v(\mathbf{r}) = -\frac{\Delta}{\pi} \operatorname{Im} G\left(E + \frac{\mathrm{i}}{2}\Gamma; \mathbf{r}, \mathbf{r}\right) = \frac{\Delta}{\pi} \sum_{n=1}^{N} [|\psi_n(\mathbf{r})|^2 + |\phi_n(\mathbf{r})|^2] \frac{\Gamma}{(E - E_n)^2 + \frac{1}{4}\Gamma^2}. \qquad (A.1)$$

Here $\psi_n(\mathbf{r})$ and $\phi_n(\mathbf{r})$ stand for the local amplitudes of the two eigenfunctions corresponding to the (double-degenerate) eigenvalue E_n. It is convenient to consider scaled eigenvalues $\epsilon_n = \pi E_n/\Delta$, defining absorption $\eta = \pi\Gamma/\Delta \equiv \frac{1}{2}\gamma$. Replacing H with $2N \times 2N$ random GSE matrix, we follow the idea first suggested in [119] and in the first step exploit the well-known fact that in the limit $N \gg 1$ eigenvector components $\psi_n(\mathbf{r})$ and $\phi_n(\mathbf{r})$ for different values of n and $\mathbf{r}$ can be treated as independent, identically distributed complex Gaussian variables. The corresponding joint probability density can be written symbolically as $\mathcal{P}_n(\psi; \phi)\mathcal{D}(\psi; \phi) = \mathrm{e}^{-N(\psi^\dagger\psi + \phi^\dagger\phi)}\mathcal{D}(\psi; \phi)$ where we introduced a shorthand notation $\psi^\dagger\psi = \sum_{n=1}^{N} |\psi_n(\mathbf{r})|^2$ (and similarly for $\phi^\dagger\phi$), with the measure $\mathcal{D}(\psi; \phi)$ being understood as the (normalized) product of differentials of independent variables.

The Laplace transform $F(s) = \int_0^\infty \mathrm{e}^{-\rho s} \mathcal{P}_\rho(\rho)\,\mathrm{d}\rho$ of the probability distribution function $\mathcal{P}_\rho$ for the normalized variable $\rho = N\pi v(\mathbf{r})$ can be then written as

$$F(s) = \left\langle \int \mathcal{P}_n(\psi; \phi)\mathcal{D}((\psi; \phi)) \prod_{n=1}^{N} \exp -\frac{sN\eta(|\psi_n(\mathbf{r})|^2 + |\phi_n(\mathbf{r})|^2)}{(\epsilon - \epsilon_n)^2 + \eta^2} \right\rangle_{\{\epsilon_n\}} \qquad (A.2)$$

where the angular brackets stand for the averaging over the joint probability density of all eigenvalues ϵ_n and the limit $N \to \infty$ is to finally taken. After performing the Gaussian integrals, we therefore arrive to the following representation:

$$F(s) = \lim_{N \to \infty} \left\langle \prod_{n=1}^{N} \frac{[(\epsilon - \epsilon_n)^2 + \eta^2]^2}{[(\epsilon - \epsilon_n)^2 + \eta^2 + 2\eta s]^2} \right\rangle_{\{\epsilon_n\}}. \tag{A.3}$$

Introducing the product $\pi_N(\lambda) = \prod_{n=1}^{N}(\lambda - \epsilon_n)$ we see that the characteristic polynomial of the GSE matrix $\hat{H}$ is simply $\pi_N^2(\lambda)$. Formula (A.3) can be conveniently rewritten in terms of this polynomial as

$$F(s) = \lim_{N \to \infty} \left\langle \frac{\pi_N^{\beta/2}(\epsilon + i\eta)\pi_N^{\beta/2}(\epsilon - i\eta)}{\pi_N^{\beta/2}(\epsilon + i\mu)\pi_N^{\beta/2}(\epsilon - i\mu)} \right\rangle_{\{\epsilon_n\}}, \qquad \mu = \sqrt{\eta^2 + 2\eta s} \tag{A.4}$$

with $\beta = 4$ for GSE. In fact, in such a form the formula retains its validity for GUE $\beta = 2$ and GOE $\beta = 1$ cases as well.

The problem of evaluating ensemble averages of products and ratios of characteristic polynomials of random matrices attracted a lot of research interest recently, both in physical and mathematical community. For $\beta = 2$, the general problem was solved in [120, 121], and the most complete set of formulae available presently for $\beta = 1, 4$ can be found in the recent paper by Borodin and Strahov [122]. When addressing the most interesting case $\beta = 1$ they were, unfortunately, able to consider only integer powers of characteristic polynomials, whereas (A.4) obviously requires knowledge of half-integer powers. In this sense the corresponding random matrix problem for $\beta = 1$ case is still outstanding. Borodin and Strahov derived the following explicit expression for the $\beta = 4$ average featuring in (A.4):

$$\lim_{N \to \infty} \left\langle \frac{\pi_N^2(\alpha_1)\pi_N^2(\alpha_2)}{\pi_N^2(\beta_1)\pi_N^2(\beta_2)} \right\rangle_{\text{GSE}} = \frac{\prod_{i,j=1}^{2}(\alpha_i - \beta_j)}{(\alpha_1 - \alpha_2)(\beta_1 - \beta_2)} \, \text{Pf} \, \hat{C} \tag{A.5}$$

assuming that $\text{Im} \, \beta_1 > 0$, $\text{Im} \, \beta_2 < 0$. Here, the matrix $\hat{C}$ has the following structure:

$$\hat{C} = \begin{pmatrix} \hat{A} & \hat{B} \\ -\hat{B}^T & \hat{D} \end{pmatrix},$$

with

$$\hat{A} = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}, \qquad \hat{D} = \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix}, \qquad \hat{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},$$

and Pf stands for the corresponding Pfaffian, $\text{Pf} \, \hat{C} = \sqrt{\det \hat{C}}$. The entries of the matrix $\hat{C}$ are given explicitly by

$$a = \frac{1}{\pi} \int_0^1 \frac{dt}{t} \sin(\alpha_1 - \alpha_2)t, \qquad d = 2\pi i \frac{\partial}{\partial \beta_1} \left(\frac{e^{i(\beta_1 - \beta_2)}}{\beta_1 - \beta_2} \right) \tag{A.6}$$

$$b_{11} = -\frac{e^{i(\beta_1 - \alpha_1)}}{\beta_1 - \alpha_1}, \qquad b_{12} = \frac{e^{i(\alpha_1 - \beta_2)}}{\alpha_1 - \beta_2}, \qquad b_{21} = -\frac{e^{i(\beta_1 - \alpha_2)}}{\beta_1 - \alpha_2}, \qquad b_{22} = \frac{e^{i(\alpha_2 - \beta_2)}}{\alpha_2 - \beta_2}. \tag{A.7}$$

Restricting our consideration in (A.4) for simplicity to $\epsilon = E = 0$ we have therefore $\alpha_1 = i\eta, \alpha_2 = -i\eta, \beta_1 = i\mu, \beta_2 = -i\mu$. Substituting this to (A.6)–(A.7) and then to (A.5), we find after straightforward computations the Laplace transform of the LDoS probability density:

$$F_{\text{GSE}}(s) = F_{\text{GUE}}(s) + \delta F(s), \tag{A.8}$$

$$F_{\text{GUE}}(s) = \frac{1}{4\eta\mu} (\mathrm{e}^{-2(\eta-\mu)}(\eta+\mu)^2 - \mathrm{e}^{-2(\eta+\mu)}(\eta-\mu)^2) \tag{A.9}$$

$$\delta F(s) = \frac{1}{4\eta\mu^2}(\eta^2 - \mu^2)^2\, \mathrm{e}^{-2\eta}\left(1 + \frac{1}{2\mu}\right)\int_0^1 \frac{\mathrm{d}t}{t}\, \sinh{(2\eta t)}. \tag{A.10}$$

We note that (i) the only *s*-dependence in the above expression comes from $\mu = \sqrt{\eta^2 + 2\eta s}$ and (ii) formula (A.9) concides with the Laplace transform of the LDoS distributuion for $\beta = 2$ case, see e.g. [88, 119].

The remaining job is to find the part $\delta F(\rho)$ of the LDoS probability density corresponding to inversion of the Laplace transform in (A.10), which gives

$$\delta F(\rho) = \frac{\mathrm{d}^2}{\mathrm{d}\rho^2}[\rho^{1/2}\, \mathrm{e}^{-\eta(\rho/2+2/\rho)}]\frac{1}{4}\sqrt{\frac{2}{\pi\eta}}\int_0^1 \frac{\mathrm{d}t}{t}\, \sinh{(2\eta t)}. \tag{A.11}$$

Knowledge of this function allows immediate restoration of the corresponding probability density (32) for the main scaling variable *x* due to relations discussed in the main body of the present paper (recall that $\eta = \gamma/2$).

Appendix B. Disordered systems of arbitrary dimension: nonlinear σ-model derivation of expressions (37)

Let $\mathbb{H}$ be a (self-adjoint) Hamiltonian describing a one-particle quantum motion in a *d*-dimensional static disordered potential. Among microscopic model Hamiltonians ensuring validity of the nonlinear sigma-model description of such a system the simplest choice seems to be Wegner's *N*-orbital model [123, 124], or its variant due to Pruisken and Schäfer [125]. Physically the models are equivalent to the so-called granulated metal model [74]. One can visualize it by considering a lattice of L^d sites (*d* standing for the spatial dimension of the sample), each site being occupied with a metallic 'granula'. The motion of a quantum particle inside each granula is assumed to be fully 'ergodic', and as such the Hamiltonian of the individual granula should be adequately modelled by a random $N \times N$ matrix $\hat{H}$ of appropriate symmetry, provided that we consider the limit $N \to \infty$. The quantum particle is also allowed to tunnel between the neighbouring granulae, the process ensuring a possibility of nontrivial diffusive motion along the lattice. Thus, the Hamiltonian $\mathbb{H}$ of the system as a whole has a form of large matrix of the size $N \times L^d$, consisting of coupled matrix blocks $\hat{H}_k, k = 1, 2, \ldots, L^d$, of the size $N \times N$. For example, for the simplest quasi-one-dimensional sample $d = 1$ such a matrix will be 'block-three-diagonal'.

Being interested in scattering, we should provide a way to incorporate description of external leads (or waveguides) attached to the sample. In the framework of the present model attaching an M_k-channel lead to the block at site *k* is done by replacing the corresponding 'intragranula' matrix block $\hat{H}_k$, with its non-self-adjoint counterpart $\hat{H}_k - \frac{\mathrm{i}}{2}\hat{\Gamma}_k$, where $N \times N$ matrix $\hat{\Gamma}_k = 2\,\mathrm{diag}\big(\kappa_1^{(k)}, \ldots, \kappa_{M_k}^{(k)}, 0, \ldots, 0\big) \geqslant 0$, and M_k staying finite when $N \to \infty$. Assuming the absorption width Γ to be uniform and identical in all the granulae, we are interested in deriving the joint probability density $\mathcal{P}(u, v)$ of real *u* and imaginary *v* parts of the local Green function in a state $|j^{(l)}\rangle$ belonging to granule at the lattice site *l*:

$$G(j^{(l)}, j^{(l)}, E_\gamma) = \langle j^{(l)}|\left(E_\gamma - \mathbb{H} + \frac{\mathrm{i}}{2}\hat{\Gamma}\right)^{-1}|j^{(l)}\rangle. \tag{B.1}$$

Following the general strategy (see section 4.1.2) we recover $\mathcal{P}(u, v)$ from the correlation function $C(z', z'')$, see (34). The calculation of the corresponding correlation function is a straightforward extension of the 'zero-dimensional' procedure employed in [40] for $\beta = 2$

and then in [84] for $\beta = 1$ (as well as for the whole crossover) to the present d-dimensional situation and will not be repeated here. The emerging supersymmetric nonlinear σ-model on the lattice is described in terms of the supermatrices Q_k, $k = 1, \ldots, L^d$, parametrized as $Q_k = -\mathrm{i}\hat{T}_k \Lambda \hat{T}_k^{-1}$ and interacting according to the 'action' (see e.g. [81])

$$S\{Q\} = -\frac{t\beta}{4} \sum_{\langle k,j \rangle}^{L^d} \mathrm{Str}\, Q_k Q_j + \frac{\beta}{4} \frac{\Gamma}{\Delta} \sum_{k}^{L^d} \mathrm{Str}\, Q_k \Lambda, \tag{B.2}$$

where the first sum goes over pairs $\langle k, j \rangle$ of nearest neighbours on the underlying lattice, Δ stands for the mean level spacing and t stands for the effective inter-granule coupling constant, which is the main control parameter of the emerging theory. We start with outlining the procedure for $\beta = 2$ symmetry when Q-matrices involved in the calculation have the smallest size 4×4. Assuming for simplicity that the granule l does not have a channel attached to it directly, the correlation function $C(z', z'')$, see (33) and (34), of resolvents of the (scaled with $\frac{N\Delta}{\pi}$) Green functions (B.1) is expressed in terms of the Q-integrals as follows

$$C_{\beta=2}(z', z'') = \prod_{k=1}^{L^d} \int \mathrm{d}\mu(Q_k) \mathcal{F}_{M_k}^{(k)}(Q_k)\, \mathrm{e}^{-S\{Q\}} \frac{\partial^2}{\partial J_1 \partial J_2} \mathrm{Sdet}^{-1}\left(U_J^{-1} + Q_l\right)\Big|_{J_1=J_2=0} \tag{B.3}$$

where the supermatrix U depends on the variables z', z'' and sources J_1, J_2 via

$$U = \tfrac{1}{2}(z' - \mathrm{i}z'')(1 + \Lambda) + \tfrac{1}{2}(z' + \mathrm{i}z'')(1 - \Lambda) + \mathrm{diag}(0, J_1, 0, J_2). \tag{B.4}$$

The 'channel factor' $\mathcal{F}_{M_k}^{(k)}(Q_k) \equiv \prod_{c=1}^{M_k} \mathrm{Sdet}^{-1}\left(1 + \mathrm{i}\kappa_c^{(k)} Q_k \Lambda\right)$ originates from coupling to continuum, where $\kappa^{(k)} = 0$ if no external channels is attached to a given granule. The factors $\mathcal{F}_M$ appearing explicitly in section 3.2 as well as in (57) are just the zero-dimensional analogue of $\mathcal{F}_{M_k}^{(k)}$. Following [81, 82], we find it convenient to introduce the function

$$Y^{(l)}(Q_l) = \prod_{k \neq l}^{L^d} \int \mathrm{d}\mu(Q_k) \mathcal{F}_{M_k}^{(k)}(Q_k)\, \mathrm{e}^{-S\{Q\}}. \tag{B.5}$$

Employing the standard Efetov-type parametrization of the matrices Q, one finds that actually $Y^{(l)}(Q)$ must be a function of only two commuting variables: $1 \leqslant \lambda_1 \leqslant \infty$ and $-1 \leqslant \lambda_0 \leqslant 1$. Then integrating out all the remaining degrees of freedom in quite a standard way (see [40] for more detail), we arrive at the representation (37) where

$$\mathcal{F}_{\beta=2}(z', z'') = \int_1^{\infty} \mathrm{d}\lambda_1 \int_{-1}^1 \frac{\mathrm{d}\lambda_0}{(\lambda_1 - \lambda_0)^2} \frac{\tilde{x} + \lambda_0}{\tilde{x} + \lambda_1} Y^{(l)}(\lambda_1, \lambda_0) \tag{B.6}$$

the main scaling variable of our theory $\tilde{x} \equiv \frac{1}{2z''}(z'^2 + z''^2 + 1)$ being introduced. This immediately verifies the structure of the important formula (37) for a general d-dimensional $\beta = 2$ system with attached scattering channels. Of course, ability to work out explicit expressions for the probability density $\mathcal{P}(u, v)$ crucially depends on the availability of the function $Y^{(l)}(\lambda_1, \lambda)$. This function has a very simple form in 'zero dimension', physically equivalent to a system consisting of a single granula:

$$Y^{(l)}(\lambda_1, \lambda_0) = \mathrm{e}^{-\gamma(\lambda_1 - \lambda_0)/2} \prod_{c=1}^{M} \frac{g_c + \lambda_0}{g_c + \lambda_1} \tag{B.7}$$

where we have assumed that the total number of external channels attached to the system is M. Each channel is characterized by its own effective coupling constant $g_c = \frac{1}{2}\left(\kappa_c + \kappa_c^{-1}\right) \geqslant 1$, the transmission coefficient being $T_c = \frac{2}{g_c+1}$. Weak-localization corrections to zero-dimensional results can be in principle found systematically following the procedures

of [104], and nonperturbative results are mainly available in quasi-1D systems, see e.g. [126], in the limit of weak absorption, $\gamma \ll 1$.

Treating systems of $\beta = 1$ symmetry class follows exactly the same steps as outlined before, although in this case the supermatrices Q are of the size 8×8 with the corresponding doubling the dimension of (B.4). Following [84, 86], we consider the whole crossover between $\beta = 1$ and $\beta = 2$ symmetry classes. Due to the TRS breaking perturbation controlled by the parameter y ($\hat{H}_k$ is no longer symmetric but still Hermitian), the action (B.2) (with $\beta = 1$) acquires additionally the symmetry breaking term $S_y\{Q\} = \frac{1}{4} y^2 \sum_k \mathrm{Str}(\tau_3 Q_k)^2$, with τ_3 being the Pauli matrix. As usual, explicit integrations are much more cumbersome and require more work, some very useful and helpful relations can be found in [86]. Exploiting the parametrization suggested there, the function $Y^{(l)}(Q)$ turns out again to be dependent on three variables $1 \leqslant \lambda_{1,2} \leqslant \infty$ and $-1 \leqslant \lambda_0 \leqslant 1$, and the analogue of (B.6) reads

$$\mathcal{F}_{\beta=1}(z', z'') = \int_1^\infty \mathrm{d}\lambda_1 \int_1^\infty \mathrm{d}\lambda_2 \int_{-1}^1 \frac{\mathrm{d}\lambda_0}{\mathcal{R}^2} \frac{\tilde{x} + \lambda_0}{\sqrt{(\tilde{x} + \tilde{\mu}_1)(\tilde{x} + \tilde{\mu}_2)}} Y^{(l)}(\lambda_1, \lambda_2, \lambda_0), \tag{B.8}$$

where $\mathcal{R} = \lambda_0^2 + \lambda_1^2 + \lambda_2^2 - 2\lambda_0\lambda_1\lambda_2 - 1$ and $\tilde{\mu}_{1,2} = \lambda_1\lambda_2 \pm \sqrt{(\lambda_1^2 - 1)(\lambda_2^2 - 1)}$, and in zero dimension:

$$Y^{(l)}(\lambda_1, \lambda_2, \lambda_0) = f(\{\lambda\})\, \mathrm{e}^{-\gamma(\lambda_1\lambda_2 - \lambda_0)/2} \prod_{c=1}^M \frac{g_c + \lambda_0}{\sqrt{(g_c + \tilde{\mu}_1)(g_c + \tilde{\mu}_2)}}. \tag{B.9}$$

Here, $f(\{\lambda\})$ coming solely due to the $\mathrm{e}^{-S_y\{Q\}}$ term is explicitly given by (39). Integration variables $\tilde{\mu}_{1,2}$ make a connection between the 'radial' parametrization of Q of [72] and the 'angular' parametrization of [2]. This will be utilized in the next appendix to get explicit result for the zero-dimensional case for $\beta = 1$.

Appendix C. Analytic continuation (40) and GOE result (44)

We prove here equivalence of the analytic continuation of the 'connected' part of $C(z', z'')$ in $z'' \to -v \pm \mathrm{i}0$ (36) to that of $\mathcal{F}(\tilde{x})$ in $\tilde{x} \to -x \pm \mathrm{i}0$ (40). Let us consider in detail first the simplest case of $\beta = 2$ symmetry class in zero dimension. Then

$$C(z', z'') = \frac{1}{z'^2 + (z'' + 1)^2} - \frac{1}{4} \left(\frac{\partial^2}{\partial z'^2} + \frac{\partial^2}{\partial z''^2} \right)$$
$$\times \int_1^\infty \mathrm{d}\lambda_1 \int_{-1}^1 \frac{\mathrm{d}\lambda_0}{\lambda_1 - \lambda_0} \frac{2z'' \, \mathrm{e}^{-\gamma(\lambda_1 - \lambda_0)/2}}{z'^2 + z''^2 + 2z''\lambda_1 + 1} \tag{C.1}$$

where we have kept in the integrand only the term remaining nonzero under the differentiation. The FT $\hat{C}(k, z'')$ with respect to the first argument reads as follows:

$$\widehat{C}(k, z'') = \frac{\pi \, \mathrm{e}^{-|k|(z''+1)}}{z'' + 1} - \frac{1}{4} \left(-k^2 + \frac{\partial^2}{\partial z''^2} \right)$$
$$\times \int_1^\infty \mathrm{d}\lambda_1 \int_{-1}^1 \frac{\mathrm{d}\lambda_0}{\lambda_1 - \lambda_0} \frac{\pi \, \mathrm{e}^{-|k|\sqrt{z''^2 + 2z''\lambda_1 + 1}}}{\sqrt{z''^2 + 2z''\lambda_1 + 1}} 2z'' \, \mathrm{e}^{-\gamma(\lambda_1 - \lambda_0)/2}. \tag{C.2}$$

Now we continue this result analytically in z'' and calculate the jump (36) on the negative real axis $z'' \to -v \pm \mathrm{i}0$, with $v > 0$. One gets readily

$$\widehat{\mathcal{P}}(k, v) = \delta(v - 1) + \frac{1}{2\pi} \left(-k^2 + \frac{\partial^2}{\partial v^2} \right)$$
$$\times \int_{\frac{1+v^2}{2v}}^\infty \mathrm{d}\lambda_1 \int_{-1}^1 \frac{\mathrm{d}\lambda_0}{\lambda_1 - \lambda_0} \frac{\cos\left(k\sqrt{2v\lambda_1 - 1 - v^2}\right)}{\sqrt{2v\lambda_1 - 1 - v^2}} v \, \mathrm{e}^{-\gamma(\lambda_1 - \lambda_0)/2}. \tag{C.3}$$

Performing now the inverse FT one arrives at

$$\mathcal{P}(u, v) = \delta(u)\delta(v - 1) + \frac{1}{4\pi}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)$$

$$\times \int_{\frac{1+v^2}{2v}}^{\infty} d\lambda_1 \int_{-1}^{1} \frac{d\lambda_0}{\lambda_1 - \lambda_0} e^{-\gamma(\lambda_1 - \lambda_0)/2}\delta(u^2 + v^2 + 1 - 2v\lambda_1)2v. \tag{C.4}$$

Making now use of $\delta(u^2 + v^2 + 1 - 2v\lambda_1)2v = \frac{\partial}{\partial\lambda_1}\Theta(2\lambda_1 v - u^2 - v^2 - 1) = \delta(\lambda_1 - x)$, with $x = \frac{1}{2v}(u^2 + v^2 + 1)$ from (28), one can immediately integrate (C.4) over λ_1, yielding $\int_{x-1}^{x+1} dt\, t^{-1} e^{-\gamma t/2}$ for the second line, which is in exact agreement with expression (43) following from the analytic continuation (40).

The proof for the $\beta = 1$ case proceeds along the same lines, explicit expressions being however more lengthy. We omit it, considering instead the less trivial derivation of expression (44). It is important to stress that for $\beta = 1$ the nonzero $F(x)$ (41) at given $x > 1$ is determined by the integration region $\mathcal{B}_x = \{(\tilde{\mu}_1, \tilde{\mu}_2)|x \leqslant \tilde{\mu}_1 < \infty, 0 \leqslant \tilde{\mu}_2 \leqslant x\}$, rather than by a single point $\lambda_1 = x$ as in the $\beta = 2$ case. It is convenient, therefore, to choose $\tilde{\mu}_{1,2}$ from (B.8) as new integration variables. Actually, they are related to those from [2] (which are already used in (16)) as follows:

$$\mu_{1,2} = \tfrac{1}{2}(\tilde{\mu}_{1,2} - 1) \equiv \tfrac{1}{2}\left[\lambda_1\lambda_2 \pm \sqrt{(\lambda_1^2 - 1)(\lambda_2^2 - 1)}\right], \qquad \mu_0 = \tfrac{1}{2}(1 - \lambda_0). \tag{C.5}$$

with the pre-exponential factor in (16) being the corresponding integration measure. Expression (41) at $y = 0$ acquires then the following form (with $w \equiv \frac{x-1}{2}$):

$$F(x) = \int_0^1 d\mu_0 \int_w^{\infty} d\mu_1 \int_0^w \frac{d\mu_2(1 - \mu_0)\mu_0(\mu_1 - \mu_2)}{\sqrt{\mu_1(1 + \mu_1)\mu_2(1 + \mu_2)}(\mu_0 + \mu_1)^2(\mu_0 + \mu_2)^2}$$

$$\times e^{-\gamma(\mu_1 + \mu_2 + 2\mu_0)/2}\frac{w + \mu_0}{\sqrt{(\mu_1 - w)(w - \mu_2)}}$$

$$= \int_0^{1/w} d\nu_0 \int_1^{\infty} d\nu_1 \int_0^1 \frac{d\nu_2(1 + \nu_0)\nu_0(\nu_1 - \nu_2)}{\sqrt{\nu_1(\nu_1 - 1)\nu_2(\nu_2 - 1)}(\nu_0 + \nu_1)^2(\nu_0 + \nu_2)^2}$$

$$\times e^{-\gamma w(\nu_1 + \nu_2 + 2\nu_0)/2}\frac{1 - w\nu_0}{\sqrt{(1 + w\nu_1)(1 + w\nu_2)}} \tag{C.6}$$

where the second equality comes after the scaling $\mu_{0,1,2} = w\nu_{0,1,2}$. It is convenient to consider now the function $\frac{d}{dx}F(x)$. An essential observation is that this function contains no contribution coming from the derivative of (C.6) on the upper limit due to vanishing of the integrand at $\nu_0 = 0, \frac{1}{w}$. We represent it in the following form:

$$\frac{dF(x)}{dx} = \int_0^{1/w} d\nu_0 \int_1^{\infty} d\nu_1 \int_0^1 \frac{d\nu_2\, e^{-\gamma w(\nu_1 + \nu_2 + 2\nu_0)/2}X(\nu_0, \nu_1, \nu_2)}{\sqrt{\nu_1(\nu_1 - 1)\nu_2(\nu_2 - 1)}(1 + w\nu_1)(1 + w\nu_2)} \tag{C.7}$$

where a rational function X contains all other terms resulting from the differentiation. It is a crucial fact that X can be further decomposed into partial fractions with respect to ν_0, yielding

$$X(\nu_0, \nu_1, \nu_2) = \sum_{i=1,2}(-1)^{i+1}\left[\frac{a_i(\gamma + d_{12})}{(\nu_0 + \nu_i)^2} + \frac{\gamma b_i + c_i d_{12}}{\nu_0 + \nu_i} + 2w\gamma\nu_i\right] \tag{C.8}$$

with functions $a_i = \nu_i(\nu_i - 1)(1 + w\nu_i)$, $b_i = 1 - 2(1 - w)\nu_i - 3w\nu_i^2$, $c_i = 1 - 2\nu_i - w\nu_i^2$ for $i = 1, 2$ and $d_{12} = [(1 + w\nu_1)(1 + w\nu_2)]^{-1}$. Substituting (C.8) in (C.7), one readily sees that integrals over ν_0 can be easily performed while remaining integrations over $\nu_{1,2}$ get completely decoupled in each term of the sum, leading finally to (44).

References

[1]   Stöckmann H J 1999 *Quantum Chaos: An Introduction* (Cambridge: Cambridge University Press)
[2]   Verbaarschot J J M, Weidenmüller H A and Zirnbauer M R 1985 *Phys. Rep.* **129** 367
[3]   Beenakker C W J 1997 *Rev. Mod. Phys.* **69** 731
[4]   Alhassid Y 2000 *Rev. Mod. Phys.* **72** 895
[5]   Doron E, Smilansky U and Frenkel A 1990 *Phys. Rev. Lett.* **65** 3072
[6]   Lewenkopf C H, Müller A and Doron E 1992 *Phys. Rev.* A **45** 2635
[7]   Schäfer R, Gorin T, Seligman T H and Stöckmann H J 2003 *J. Phys. A: Math. Gen.* **36** 3289
[8]   Méndez-Sánchez R A, Kuhl U, Barth M, Lewenkopf C H and Stöckmann H J 2003 *Phys. Rev. Lett.* **91** 174102
[9]   Schanze H, Stöckmann H J, Martínez-Mares M and Lewenkopf C H 2005 *Phys. Rev.* E **71** 016223
[10] Kuhl U, Martínez-Mares M, Méndez-Sánchez R A and Stöckmann H J 2005 *Phys. Rev. Lett.* **94** 144101
[11] Barthélemy J, Legrand O and Mortessagne F 2005 *Europhys. Lett.* **70** 162
[12] Lobkis O I, Rozhkov I S and Weaver R L 2003 *Phys. Rev. Lett.* **91** 194101
[13] Hul O, Bauch S, Pakonski P, Savytskyy N, Zyczkowski K and Sirko L 2004 *Phys. Rev.* E **69** 056205
[14] Kogan E, Mello P A and Liqun H 2000 *Phys. Rev.* E **61** R17
[15] Beenakker C W J and Brouwer P W 2001 *Physica* E **9** 463
[16] Savin D V and Sommers H J 2003 *Phys. Rev.* E **68** 036211
[17] Fyodorov Y V 2003 *JETP Lett.* **78** 250
[18] Savin D V and Sommers H J 2004 *Phys. Rev.* E **69** 035201(R)
[19] Rozhkov I, Fyodorov Y V and Weaver R L 2003 *Phys. Rev.* E **68** 016204
[20] Rozhkov I, Fyodorov Y V and Weaver R L 2004 *Phys. Rev.* E **69** 036206
[21] Fyodorov Y V and Savin D V 2004 *JETP Lett.* **80** 725
[22] Savin D V, Sommers H-J and Fyodorov Y V 2005 *JETP Lett.* **82** 603 (*Preprint* cond-mat/0502359)
[23] Martinez-Mares M and Mello P A 2005 *Phys. Rev.* E **72** 026224 *Preprint* cond-mat/0504150
[24] Fyodorov Y V and Ossipov A 2004 *Phys. Rev. Lett.* **92** 084103
[25] Hemmady S, Zheng X, Ott E, Antonsen T M and Anlage S M 2005 *Phys. Rev. Lett.* **94** 014102
[26] Hemmady S, Zheng X, Antonsen T M, Ott E and Anlage S M 2005 *Phys. Rev.* E **71** 056215
[27] Zheng X, Antonsen T M and Ott E 2004 *Preprint* cond-mat/0408327
[28] Zheng X, Antonsen T M and Ott E 2004 *Preprint* cond-mat/0408317
[29] Mirlin A D 2000 *Phys. Rep.* **326** 259
[30] Fyodorov Y V and Alhassid Y 1998 *Phys. Rev.* A **58** R3375
[31] Sokolov V V, Rotter I, Savin D V and Müller M 1997 *Phys. Rev.* C **56** 1044
[32] Gorin T, Mehlig B and Ihra W 2004 *J. Phys. A: Math. Gen.* **37** L345
[33] Misirpashaev T Sh, Brouwer P W and Beenakker C W J 1997 *Phys. Rev. Lett.* **79** 1841
[34] Savin D V, Fyodorov Y V and Sommers H J 2005 *Preprint* nlin.CD/0506040
[35] Ossipov A and Fyodorov Y V 2005 *Phys. Rev.* B **71** 125133
[36] Livšic M S 1957 *Sov. Phys.—JETP* **4** 91
      Livšic M S 1973 *Operators, Oscillations, Waves (Open Systems)* (*Transl. Math. Monographs* vol 34) (Providence, RI: American Mathematical Society)
[37] Mahaux C and Weidenmüller H A 1969 *Shell-model Approach to Nuclear Reactions* (Amsterdam: North-Holland)
[38] Sokolov V V and Zelevinsky V G 1989 *Nucl. Phys.* A **504** 562
[39] Lewenkopf C H and Weidenmüller H A 1991 *Ann. Phys., NY* **212** 53
[40] Fyodorov Y V and Sommers H J 1997 *J. Math. Phys.* **38** 1918
[41] Dittes F M 2000 *Phys. Rep.* **339** 215
[42] Nussenzveig H M 1972 *Causality and Dispersion Relations* (New York: Academic)
[43] Pichugin K, Schanz H and Šeba P 2001 *Phys. Rev.* E **64** 056227
[44] Stöckmann H J, Persson E, Kim Y H, Barth M, Kuhl U and Rotter I 2002 *Phys. Rev.* E **65** 066211
[45] Savin D V, Sokolov V V and Sommers H J 2003 *Phys. Rev.* E **67** 026215
[46] Barthélemy J, Legrand O and Mortessagne F 2005 *Phys. Rev.* E **71** 016205
[47] Wigner E 1955 *Phys. Rev.* **98** 145
[48] Smith F T 1960 *Phys. Rev.* **118** 349
[49] Lehmann N, Savin D V, Sokolov V V and Sommers H J 1995 *Physica* D **86** 572
[50] Sokolov V V and Zelevinsky V 1997 *Phys. Rev.* C **56** 311
[51] Lyuboshitz V L 1977 *Phys. Lett.* B **72** 41
[52] Muga J G and Leavens C R 2000 *Phys. Rep.* **338** 353
[53] de Carvalho C A A and Nussenzveig H M 2002 *Phys. Rep.* **364** 83

[54] Mehta M L 1991 *Random Matrices* 2nd edn (New York: Academic)
[55] Guhr T, Müller-Groeling A and Weidenmüller H A 1998 *Phys. Rep.* **299** 189
[56] Lehmann N, Saher D, Sokolov V V and Sommers H J 1995 *Nucl. Phys.* A **582** 223
[57] Brouwer P W and Beenakker C W J 1997 *Phys. Rev.* B **55** 4695
[58] Sokolov V V 2004 *Preprint* cond-mat/0409690
[59] Lobkis O I, Weaver R L and Rozhkov I S 2000 *J. Sound Vib.* **237** 281
[60] Muga J, Palao J, Navarro B and Egusquiza I 2004 *Phys. Rep.* **395** 357
[61] Ramakrishna S A and Kumar N 2000 *Phys. Rev.* B **61** 3163
[62] Davy J L 1987 *J. Sound Vib.* **115** 145
[63] Alhassid Y and Levine R D 1992 *Phys. Rev.* A **46** 4650
[64] Alhassid Y and Fyodorov Y V 1998 *J. Phys. Chem.* A **102** 9577
[65] Zheng X, Hemmady S, Antonsen T M, Anlage S M and Ott E 2005 *Preprint* cond-mat/0504196
[66] Verbaarschot J J M 1986 *Ann. Phys., NY* **168** 368
[67] Sommers H J, Fyodorov Y V and Titov M 1999 *J. Phys. A: Math. Gen.* **32** L77
[68] Fyodorov Y V and Khoruzhenko B A 1999 *Phys. Rev. Lett.* **83** 65
[69] Fyodorov Y V and Sommers H J 2003 *J. Phys. A: Math. Gen.* **36** 3303
[70] Fyodorov Y V and Mehlig B 2002 *Phys. Rev.* E **66** 045202(R)
[71] Gorin T and Seligman T H 2002 *Phys. Rev.* E **65** 026214
[72] Efetov K B 1983 *Adv. Phys.* **32** 53
[73] Pluhař Z, Weidenmüller H A, Zuk J A, Lewenkopf C H and Wegner F J 1995 *Ann. Phys., NY* **243** 1
[74] Efetov K B 1996 *Supersymmetry in Disorder and Chaos* (Cambridge: Cambridge University Press)
[75] Savin D V and Sokolov V V 1997 *Phys. Rev.* E **56** R4911
[76] Blatt J M and Weisskopf V F 1979 *Theoretical Nuclear Physics* (Berlin: Springer)
[77] Hauser W and Feshbach H 1952 *Phys. Rev.* **87** 366
[78] Moldauer P A 1975 *Phys. Rev.* C **11** 426
[79] Agassi D, Weidenmüller H A and Mantzouranis G 1975 *Phys. Rep.* **22** 146
[80] Friedman W A and Mello P A 1985 *Ann. Phys., NY* **161** 276
[81] Mirlin A D and Fyodorov Y V 1994 *Phys. Rev. Lett.* **72** 526
[82] Mirlin A D and Fyodorov Y V 1994 *J. Phys.* I **4** 655
[83] Gopar V A, Mello P A and Büttiker M 1996 *Phys. Rev. Lett.* **77** 3005
[84] Fyodorov Y V, Savin D V and Sommers H J 1997 *Phys. Rev.* E **55** R4857
[85] Dupius N and Montambaux G 1991 *Phys. Rev.* B **43** 14390
[86] Altland A, Iida S, Müller-Groeling A and Weidenmüller H A 1992 *Ann. Phys., NY* **219** 148
[87] Pandey A and Mehta M L 1983 *Commun. Math. Phys.* **16** 2655
[88] Efetov K B and Prigodin V N 1993 *Phys. Rev. Lett.* **70** 1315
[89] Taniguchi N and Prigodin V N 1996 *Phys. Rev.* B **54** R14305
[90] Mello P A 1995 Mesoscopic quantum physics *Proc. of the Les-Houches Summer School, Session LXI*
     ed E Akkermans, G Montambaux, J L Pichard and J Zinn-Justin (Amsterdam: Elsevier) p 435
[91] Brouwer P W, Frahm K M and Beenakker C W J 1997 *Phys. Rev. Lett.* **78** 4737
[92] Brouwer P W, Frahm K M and Beenakker C W J 1999 *Waves Random Media* **9** 91
[93] Schomerus H, van Bemmel K J H and Beenakker C W J 2000 *Europhys. Lett.* **52** 518
[94] Gopar V A and Mello P A 1998 *Europhys. Lett.* **42** 131
[95] Savin D V, Fyodorov Y V and Sommers H J 2001 *Phys. Rev.* E **63** 035202(R)
[96] Sommers H J, Savin D V and Sokolov V V 2001 *Phys. Rev. Lett.* **87** 094101
[97] Glück M, Kolovsky A R and Korsch H J 1999 *Phys. Rev. Lett.* **82** 1534
[98] Kottos T and Smilansky U 2003 *J. Phys. A: Math. Gen.* **36** 3501
[99] Akguc G and Reichl L E 2003 *Phys. Rev.* E **67** 046202
[100] Genack A Z, Sebbah P, Stoytchev M and van Tiggelen B A 1999 *Phys. Rev. Lett.* **82** 715
[101] Chabanov A A and Genack A Z 2001 *Phys. Rev. Lett.* **87** 233903
[102] Jian Z, Pearce J and Mittleman D M 2003 *Phys. Rev. Lett.* **91** 033903
[103] Pearce J, Jian Z and Mittleman D M 2003 *Phys. Rev. Lett.* **91** 043903
[104] Fyodorov Y V and Mirlin A D 1995 *Phys. Rev.* B **51** 13403
[105] Méndez-Bermúndez J A and Kottos T 2005 *Phys. Rev.* B **72** 064108
[106] Zirnbauer M R 1986 *Phys. Rev.* B **34** 6394
[107] Mello P A, Pereyra P and Seligman T 1985 *Ann. Phys.* **161** 254
[108] Brouwer P W 1995 *Phys. Rev.* B **51** 16878
[109] Beenakker C W J 1998 *Phys. Rev. Lett.* **81** 1829
[110] Beenakker C W J 1999 *Diffusive Waves in Complex Media (NATO ASI Series C* vol 531*)* ed J P Fouque
     (Dordrecht: Kluwer) p 137

[111] Mandel L and Wolf E 1995 *Optical Coherence and Quantum Optics* (Cambridge: Cambridge University Press)
[112] Paasschens J C J, Misirpashaev T Sh and Beenakker C W J 1996 *Phys. Rev.* B **54** 11887
[113] Hackenbroich G, Viviescas C, Elattari B and Haake F 2001 *Phys. Rev. Lett.* **86** 5262
[114] Cao H 2003 *Waves Random Media* **13** R1
[115] Schomerus H, Frahm K M, Patra M and Beenakker C W J 2000 *Physica* A **278** 469
[116] Hackenbroich G, Viviescas C and Haake F 2002 *Phys. Rev. Lett.* **89** 083902
[117] Altland A and Zirnbauer M R 1997 *Phys. Rev.* B **55** 1142
[118] Beenakker C W J 2005 *Lect. Notes Phys.* **667** 131
[119] Beenakker C W J 1994 *Phys. Rev.* B **50** 15170
[120] Fyodorov Y V and Strahov E 2003 *J. Phys. A: Math. Gen.* **36** 3203
[121] Strahov E and Fyodorov Y V 2003 *Commun. Math. Phys.* **241** 343
[122] Borodin A and Strahov E 2004 *Preprint* math-ph/0407065 (*Commun. Pure Appl. Math.* to appear)
[123] Wegner F 1979 *Z. Phys.* B **35** 207
[124] Schäfer L and Wegner F 1980 *Z. Phys.* B **38** 113
[125] Pruisken A M M and Schäfer L 1982 *Nucl. Phys.* B **200** 20
[126] Fyodorov Y V and Mirlin A D 1994 *Int. J. Mod. Phys.* **8** 3795